REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED		1b. RESTRICTIVE MARKINGS	
2a. SECURITY CLASSIFICATION AUTHORITY		3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution is unlimited.	
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE			
4. PERFORMING ORGANIZATION REPORT NUMBER(S) TR 6929A		5. MONITORING ORGANIZATION REPORT NUMBER(S)	
6a. NAME OF PERFORMING ORGANIZATION Naval Underwater Systems Center	6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION	
6c. ADDRESS (City, State, and ZIP Code). New London Laboratory New London, CT 06320		7b. ADDRESS (City, State, and ZIP Code)	
8a. NAME OF FUNDING/SPONSORING ORGANIZATION	8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER	

8c. ADDRESS (City, State, and ZIP Code)	10. SOURCE OF FUNDING NUMBERS			
	PROGRAM ELEMENT NO. 61152N	PROJECT NO. A75205	TASK NO.	WORK UNIT ACCESSION NO.

11. TITLE (Include Security Classification)
Operating Characteristics of an Or-ing and Selection Processor with Pre- and Post-Averaging

12. PERSONAL AUTHOR(S)
Albert H. Nuttall

13a. TYPE OF REPORT	13b. TIME COVERED FROM ___ TO ___	14. DATE OF REPORT (Year, Month, Day) 1987 JANUARY 7	15. PAGE COUNT

16. SUPPLEMENTARY NOTATION
TR 6929A supersedes TR 6929

17.	COSATI CODES		18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	Operating Characteristics Post-Averaging
			Or-ing and Selection Detection Probability
			Pre-Averaging False Alarm Probability

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

The operating characteristics, namely detection probability vs. false alarm probability, are derived and evaluated for a multi-channel input to an or-ing and selection processor with pre- and post-averaging. The input signal-to-noise ratio, the number of input channels, the amount of pre-averaging prior to or-ing, and the amount of post-averaging after selection can be varied in order to investigate the tradeoffs between the various parameters. A wide variety of results are plotted, along with simulation results that verify the analysis.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT ☑ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED	
22a. NAME OF RESPONSIBLE INDIVIDUAL Albert H. Nuttall	22b. TELEPHONE (Include Area Code) (203) 440-4618	22c. OFFICE SYMBOL Code 3314

DD FORM 1473, 84 MAR   83 APR edition may be used until exhausted.   SECURITY CLASSIFICATION OF THIS PAGE
All other editions are obsolete.

18. SUBJECT TERMS (CONT'D.)

    Multi-Channel Input
    Simulation
    Characteristic Function
    Effective Number of Independent Samples
    Gaussian Inputs
    Greatest-Of Device

Operating Characteristics of an Or-ing And Selection Processor With Pre- and Post-Averaging

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Preface

When this report was originally published as TR 6929 on 4 May 1983, it contained information proprietary to Bell Laboratories. Its distribution status has now been revised and the material is being published for public release as TR 6929A, dated 7 January 1987, to reflect the fact that U. S. Patent Number 4622519 has been assigned to AT&T Bell Laboratories, regarding this selection processor with pre- and post-averaging.

Reviewed and Approved: 7 January 1987
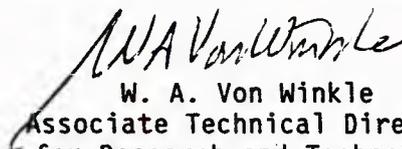
W. A. Von Winkle
Associate Technical Director
for Research and Technology

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LIST OF SYMBOLS

N	number of input channels
K	amount of pre-averaging
M	amount of post-averaging
h	hypothesis number: 0 or 1
H_0	signal absent hypothesis
H_1	signal present hypothesis
$x_n(t)$	channel n input at time t
$f_x^{(h)}(\xi)$	input characteristic function
$a_n(t)$	pre-averager n output at time t
$y(t)$	output of Greatest-Of
$\ell(t)$	channel number of largest input to Greatest-Of, at time t
$v(t)$	selector output at time t
$w(t)$	post-averager output at time t; decision variable
Λ	threshold value
P_D	detection probability
P_F	false alarm probability
$\mu_x^{(h)}$	mean of input under hypothesis h
σ_x^2	variance of input under both hypotheses
d	input deflection: $(\mu_x^{(1)} - \mu_x^{(0)})/\sigma_x$
γ	modified threshold; (C-12)
I_ℓ	effective number of independent samples; (A-9)-(A-12)
I_2	integer part of I_ℓ
α_2, β_2	parameters associated with I_ℓ and I_2: (C-16) and (B-3)
Q_a	probability that signal channel dominates Greatest-Of on one trial; (C-5)-(C-7)
T_t	total number of trials in simulation
w_k	decision variable for k-th trial

OPERATING CHARACTERISTICS OF AN OR-ING AND SELECTION
PROCESSOR WITH PRE- AND POST-AVERAGING

INTRODUCTION

The analysis of systems with or-ing has been the subject of considerable interest [1-8]. These analyses have been in terms of a second-moment theory and have usually made timely use of the Central Limit Theorem at various points in the system, such as an accumulator output. A notable exception to this approach is the exact analysis in [9], where the detection and false alarm probabilities for a system with quantizers, or-ing, and an accumulator were determined exactly. There, the characteristic function of the decision variable was determined exactly via a fast Fourier transform to yield threshold-crossing probabilities.

Here, we attempt to continue this theme of getting exact detection and false alarm probabilties for an or-ing system with pre- and post-averaging. However, the analytical situation is greatly complicated by the presence of the pre- and post-averagers: the pre-averager creates statistical dependencies between its successive output time samples that the post-averager must operate on. For the signal present hypothesis H_1, we thus encounter the classical problem of determining the output distribution function of a linear system subject to non-Gaussian and non-white noise; since this is an unsolved problem, we must instead revert to determining an approximation for the characteristic function and/or cumulative distribution function of the decision variable. On the other hand, for the signal absent hypothesis H_0, the characteristic function of the decision variable can be determined in closed form. Thus we have a mixed situation, where part of the analysis is exact and part approximate.

The technique utilized here is not a second-moment approach, although one part of the analysis does match moments through second-order, in order to determine the effective number of independent samples at the or-ing input. The utility of this combined approximate analysis is borne out by simulation which corroborates the technique in a wide variety of examples evaluated numerically.

The inclusion of or-ing in a system in the usual fashion necessitates an increase in the decision threshold, if the false alarm probability is to be kept the same as when there was no or-ing, i.e., one input channel. The amount of threshold increase required is a function of the number of input channels, N, and must therefore be re-calculated every time N is changed. Also, this increased threshold leads to decreased detection probabilities, which is undesirable.

Here we will investigate a novel or-ing and selection processor proposed* by Bell Laboratories which uses a threshold completely independent of the number of input channels, N, and yet also realizes a false alarm probability independent of N. The required threshold does, of course, depend on the required false alarm probability and the statistics of the noise-only on one input channel, in addition to the amount of post-averaging; however, it is independent of N and the amount of pre-averaging.

Although the new processor utilizes a threshold independent of N, it still suffers a loss in detection probability relative to no or-ing, because a succession of decisions must be reached after the pre-averager (i.e., in the or-ing device) but before the post-averager. That is, the nonlinear operations are not deferred until after all the linear signal-enhancing operations are completed, but in fact, occur at an intermediate point in the processor. How well the new processor performs in comparison with a standard or-ing system with pre- and post-averaging is not known; the latter is currently under investigation and will be reported on separately.

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*U.S. Patent Number 4622519 has been assigned to AT&T Bell Laboratories, regarding this selection processor with pre- and post-averaging.

DESCRIPTION OF OR-ING AND SELECTION PROCESSOR

The system of interest is depicted in figure 1. There are N input channels, and the input on the n-th channel at discrete time t is $x_n(t)$. The inputs $\{x_n(t)\}$ are stationary and statistically independent of each other, from channel-to-channel, and from one time instant to any other within the same channel. Furthermore, the channels without a signal in them are identically distributed, with an arbitrary characteristic function $f_x^{(0)}(\xi)$, where the superscript denotes hypothesis H_0, signal absent from all channels. For hypothesis H_1, the signal is present in one and only one (unknown) channel, and remains in that channel for the total observation time leading to a particular decision at the system output $w(t)$; this total time is K+M samples for the system of figure 1. The characteristic function of the signal-bearing input channel is $f_x^{(1)}(\xi)$. It is not presumed that the signal and noise are additive at the inputs $\{x_n(t)\}$; all that is needed are the input characteristic functions $f_x^{(h)}$ for hypotheses h=0 and 1. (Later, for numerical evaluation purposes, we will specialize the results to Gaussian inputs.)

The inputs $\{x_n(t)\}$ are all delayed by one unit of time (for reasons to be seen below) and then subjected to pre-averaging of K samples; that is, the pre-averager outputs are

$$a_n(t) = \sum_{k=1}^{K} x_n(t-k) \quad \text{for} \quad 1 \le n \le N \quad . \tag{1}$$

Due to the unit delay, $a_n(t)$ is the sum of the past K samples of input $x_n(t)$ on channel n. The purpose of this pre-averaging is to enhance the signal (if present), prior to making a nonlinear decision in the or-ing procedure.

The greatest-of device has two outputs. The first is the actual largest input it is subjected to at time t:

$$y(t) = \max\{a_1(t), \ldots, a_N(t)\} \equiv \max_{1 \le n \le N} \{a_n(t)\} \equiv a_{\ell(t)}(t) . \tag{2}$$

3
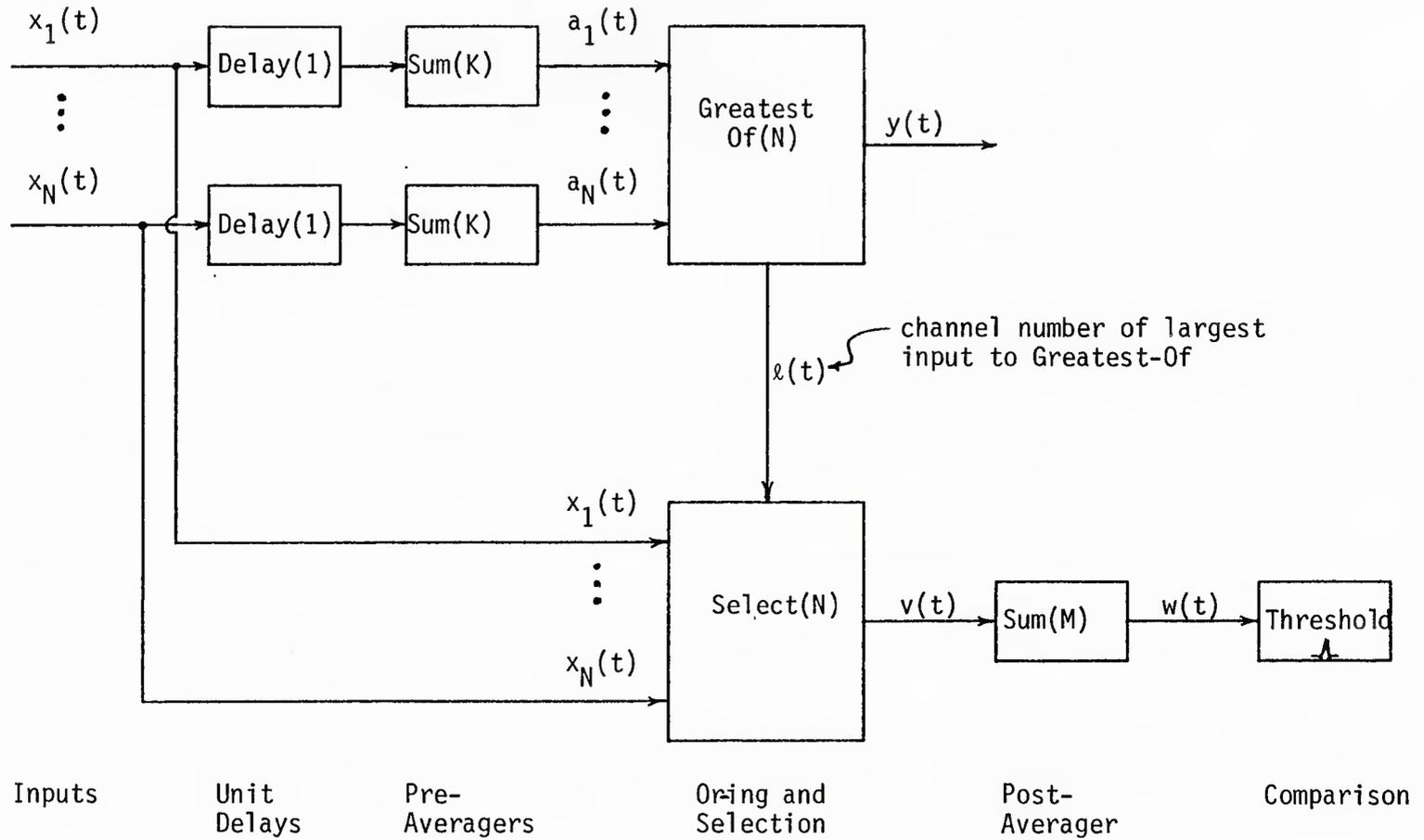
Figure 1. Or-ing and Selection Processor with Pre- and Post-Averaging

The other output is $\ell(t)$, as indicated in (2); it is the channel number at time t of the largest input to the greatest-of device. Although y(t) is discarded, $\ell(t)$ is used in the selection procedure.

The channel number output, $\ell(t)$, is used to make a selection of one of the N inputs <u>at time t</u>, not t-1. That is, the selector output is

$$v(t) = x_{\ell(t)}(t) \equiv \tilde{x}(t;\ell(t)), \qquad (3)$$

which is one of the current input samples.

Finally, the post-averager output is the sum of M consecutive selector outputs:

$$w(t) = \sum_{j=0}^{M-1} v(t-j) = \sum_{j=0}^{M-1} \tilde{x}(t-j;\ell(t-j)) \quad . \qquad (4)$$

This quantity is called the decision variable; it is compared with a threshold λ for decisions about signal absence vs signal presence in some input. It is important to observe that although the decision variable only employs M samples, the "oldest" of these samples itself depends on the previous K input samples, by way of the pre-averaging. Thus if the processor is to function well under hypothesis H_1, the signal must be present at the input for a total of K+M samples, not just M. This feature must be kept in mind when comparing this processor with others, in terms of the detection probability and false alarm probabilities; both processors must be allowed K+M samples for a fair comparison.

We recall from (1) that the pre-averager outputs at time t were unable to utilize the current input samples $\{x_n(t)\}$, for $1 \leq n \leq N$. Since the system inputs are all independent at different times, the selection procedure will always yield a random variable independent of those used in the or-ing. For H_0, this means that post-averager output w(t) in (4) is the sum of M independent, identically distributed, random variables, each with characteristic function $f_X^{(0)}(\xi)$, <u>regardless</u> of the particular sequence of channel values $\ell(t)$, $\ell(t-1)$, ..., $\ell(t-M+1)$, that occurred at the selector

5

input. Thus the statistics of w(t) under H_0 are independent of N, the number of input channels, but depend solely on $f_X^{(0)}(\xi)$ and M. This enables the setting of a threshold which realizes a fixed false alarm probability , regardless of the value of N.

Under H_1, although the post-averager output is the sum of M statistically independent random variables,* some of the selector outputs are drawn from the actual signal channel (if the pre-averager is functioning well), but some are noise-only channel inputs. The number of correct signal-channel selections can be anywhere in the range (0,M); this greatly complicates the analysis for the detection probability, and necessitates some approximations.

Behavior of Pre-Averager Outputs

Since the memory of the pre-averagers is K samples, the inputs to the greatest-of device cannot "forget" the past very quickly. What this means is that if one system input is momentarily very noisy, it continues to adversely affect or-ing decisions for a considerable time. Thus, channel-number output $\ell(t)$ can remain at the same value for several consecutive times, <u>even if the signal is absent</u>. A sample plot of this behavior is depicted in figure 2 for K=10 and N=5. Thus, for example, one particular channel pre-averager output dominates from sample t=13 to t=27.

When the signal is added, the situation improves, in that the signal channel dominates most often; see figure 3. However, there are still extended regions, such as (16,26) and (68,90), where one of the noise channels dominates. The effects of these extended regions of dominance must be accounted for in any analysis of performance.

There are conflicting requirements at work in the processor of figure 1. One indicates that K should be increased so that good decisions can be reached in the greatest-of. On the other hand, larger K can cause prolonged dominance, as indicated above; and it subtracts from the time, M, available for post-averaging, if total observation time K+M is fixed or limited. Since it is also desirable to increase M, for better quality decisions at the system

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* when conditioned upon a particular sequence of $\ell(t)$ values; see (A-1), (A-5), and (A-6).

$\{a_n(t)\}$
for
N=5

Time Sample t
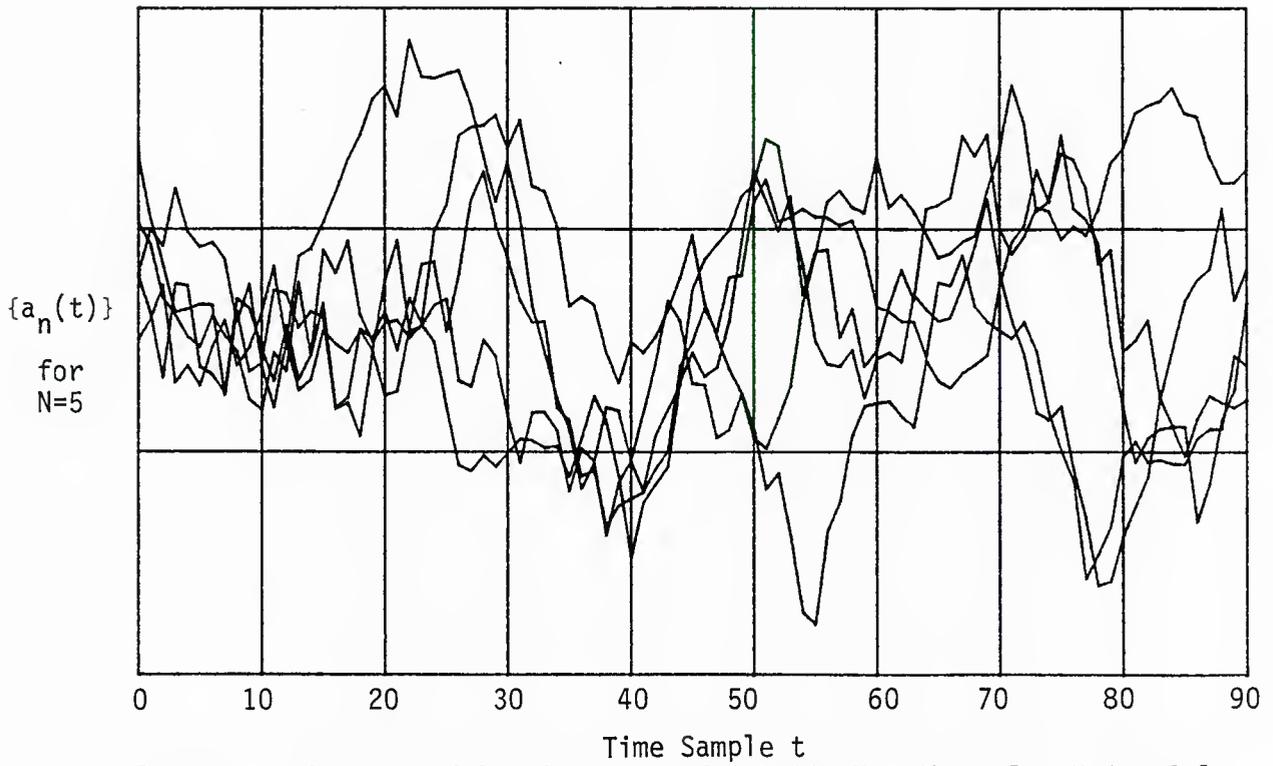
Figure 2.   Outputs of Pre-Averagers for K=10, N=5 Channels, Noise-Only

$\{a_n(t)\}$
for
N=5

Time Sample t
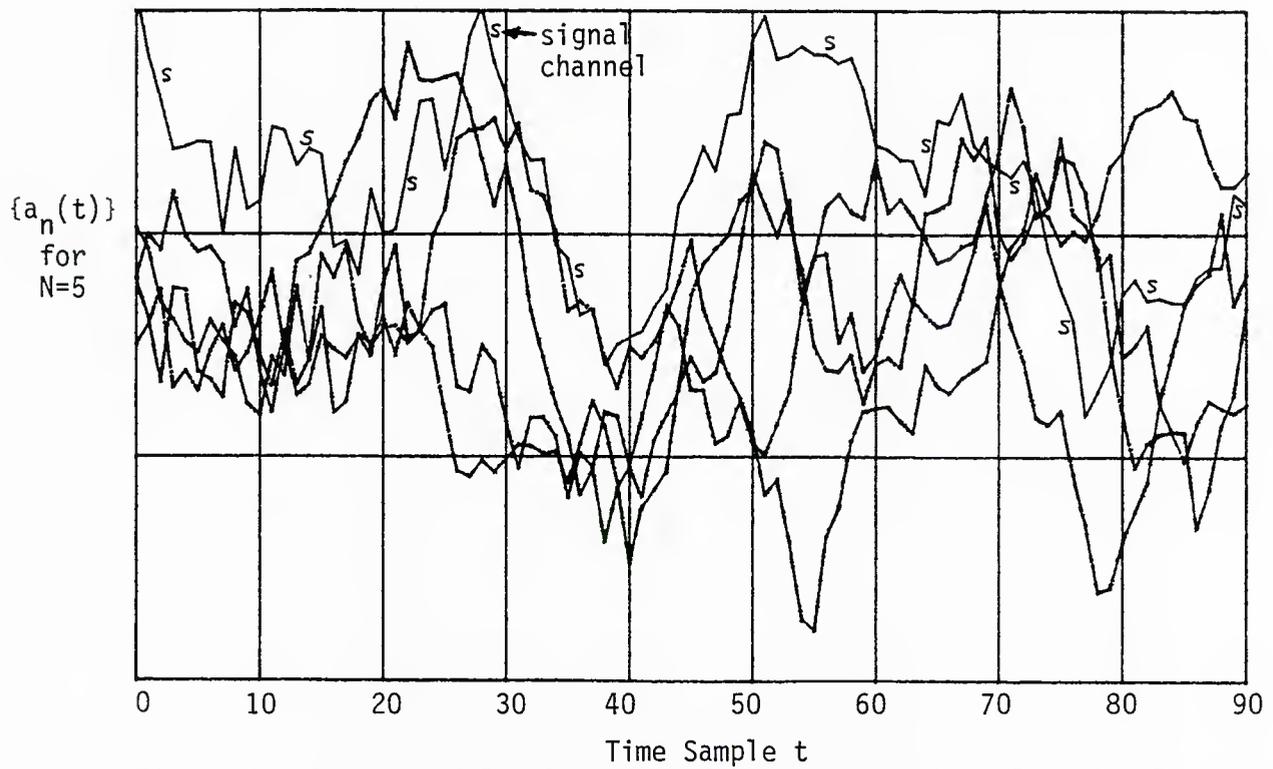
Figure 3.   Outputs of Pre-Averagers for K=10, N=5 Channels, Signal Present

output, it is conceivable that there is an optimum division between pre- and post-averaging times, if K+M is fixed. This behavior can be extracted from the operating characteristics, when plotted for a wide variety of values of K and M. Of course, there remains the dependency of any conclusions on N and the input statistics.

RESULTS

Derivation of detection probability P_D and false alarm probability P_F is conducted in a series of appendices, under various degrees of generality. The only one we consider numerically here is that of Gaussian inputs, for which the input characteristic functions are

$$f_X^{(h)}(\xi) = \exp(i\xi\mu_X^{(h)} - \frac{1}{2}\xi^2\sigma_X^2) \quad \text{for} \quad h = 0 \text{ and } 1. \tag{5}$$

Thus the two hypotheses have different means $\mu_X^{(h)}$, but the same variance σ_X^2, for the system inputs. We define the system input deflection

$$d \equiv \frac{\mu_X^{(1)} - \mu_X^{(0)}}{\sigma_X} . \tag{6}$$

The detection probability and false alarm probability are given in (C-19) and (C-20), where the various parameters are also explained. In the derivations, it was necessary to define the number of independent samples at a pre-averager output in an interval of length M, and then to approximate characteristic functions raised to fractional powers. The details are lengthy and are reserved for the appendices for the interested reader.

A variety of cases are plotted in the following figures. Values of the parameters have been chosen as indicated in table 1 for figures 4 – 35. We have looked at all possible combinations of these choices, for a total of 4*4*2 = 32 cases.

N, Number of Input Channels	K, Amount of Pre-Averaging	M, Amount of Post-Averaging
2	1	32
4	2	64
8	4	
16	8	

Table 1. Values of N, K, M Considered for Figures 4 – 35

Superposed on most figures is one simulation run for a representative value of d, equal to one utilized in the analysis; for example, figure 4 contains a simulation result for d = .8. These (jagged) simulation curves closely overlay the (smooth) analysis curves except in the region of rare events, namely where the detection probability is near unity $(1-P_D << 1)$; the random discrepancy in this latter region is due to the relatively small and limited number of trials employed in the simulation to estimate P_D, namely 1000 trials. There is no need to simulate the false alarm case at all, since we can calculate the false alarm probability exactly in any region and for any parameter values.

For the smaller values of N, the characteristics are nearly equally spaced; however, as N increases, the curves get more crowded around the d=0 curve. This is a manifestation of small signal suppression, brought about by the nonlinear action of the or-ing operation. The plots in figures 4 - 35 are also generally fairly straight; this is due to the considerable amount of post-averaging used, namely M = 32 or 64.

When M is decreased, as in figures 36 - 39, the operating characteristics develop a significant knee, where the detection probability temporarily saturates as the false alarm probability increases. However, this undesirable behavior generally occurs for the larger values of false alarm probability, where the system will not be operated. The superposed simulation results confirm the presence of this knee in the operating characteristics.

SUMMARY

The detection and false alarm probabilities have been derived for the or-ing and selection processor with pre- and post-averaging and specialized to Gaussian inputs. Plots of the operating characteristics for a variety of choices of number of input channels, amount of pre-averaging, amount of post-averaging, and input signal strength are given. Superposed on most of the analytical results is a simulation result for each case, which corroborates the approximations which were necessary for a tractable analysis. Other cases of interest to the reader may be obtained by means of the program listed in appendix D in BASIC for the Hewlett Packard 9845 Desk Calculator.
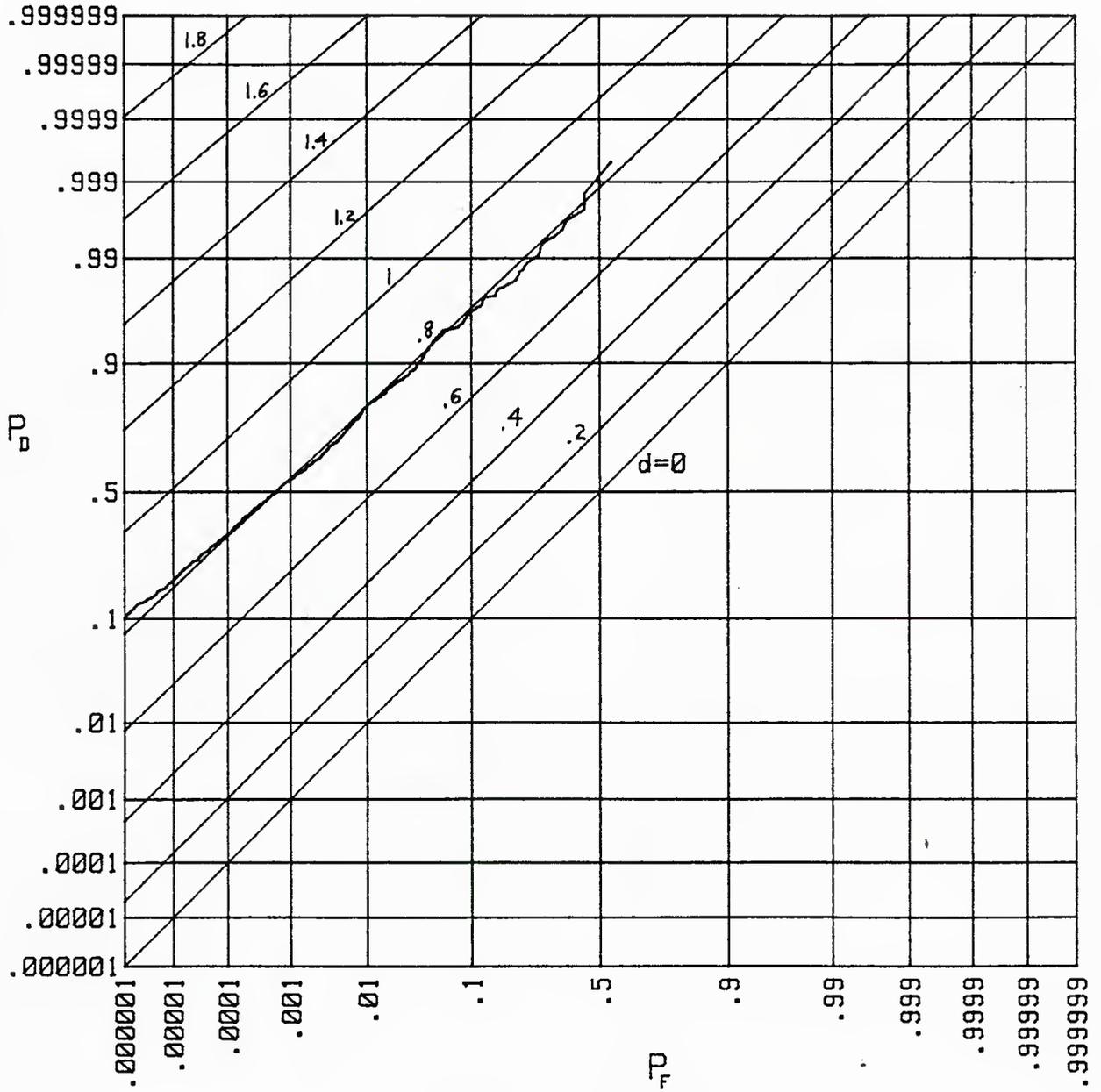
Figure 4. Operating Characteristics for N=2, K=1, M=32
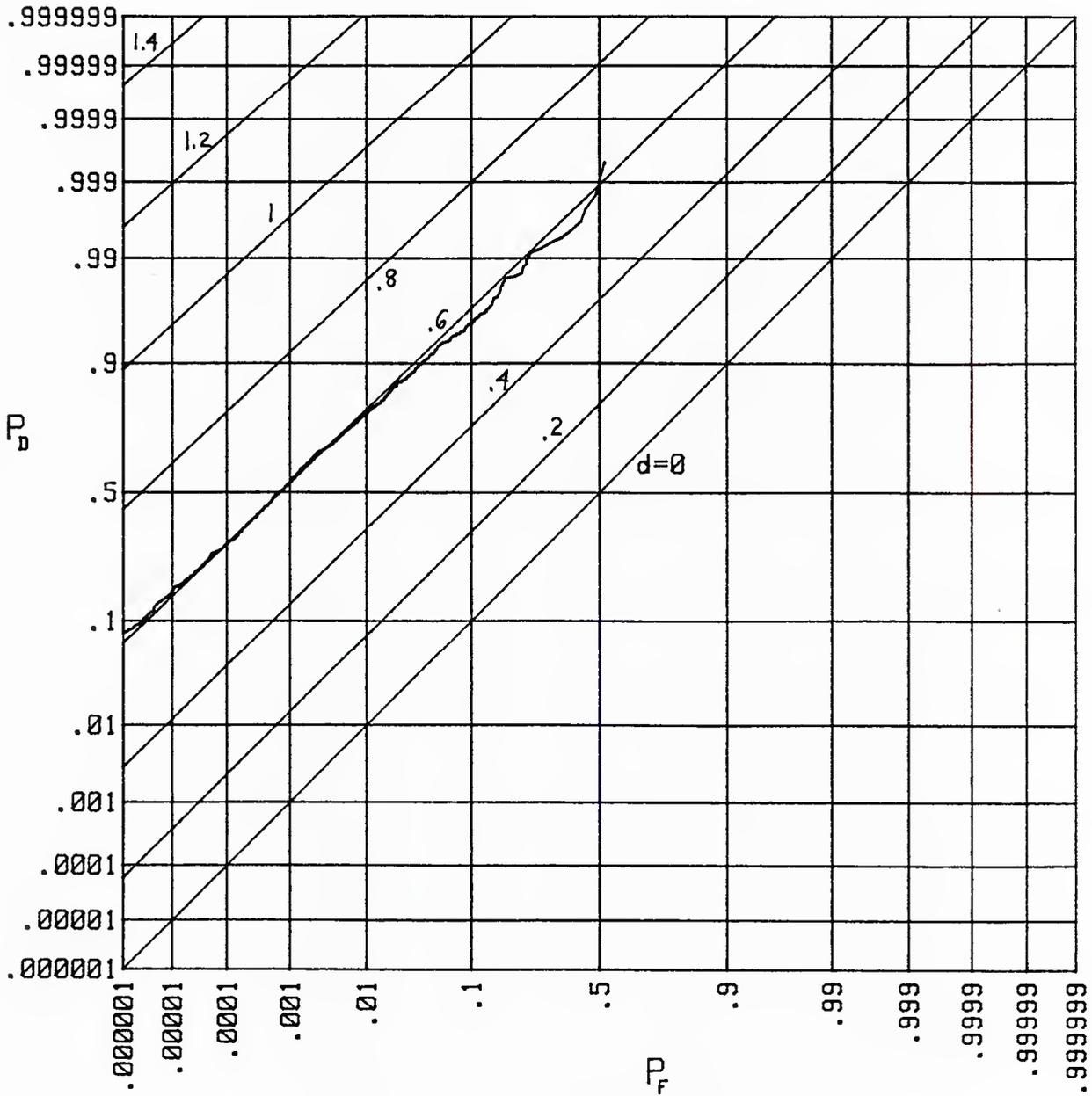
Figure 5.   Operating Characteristics for N=2, K=1, M=64
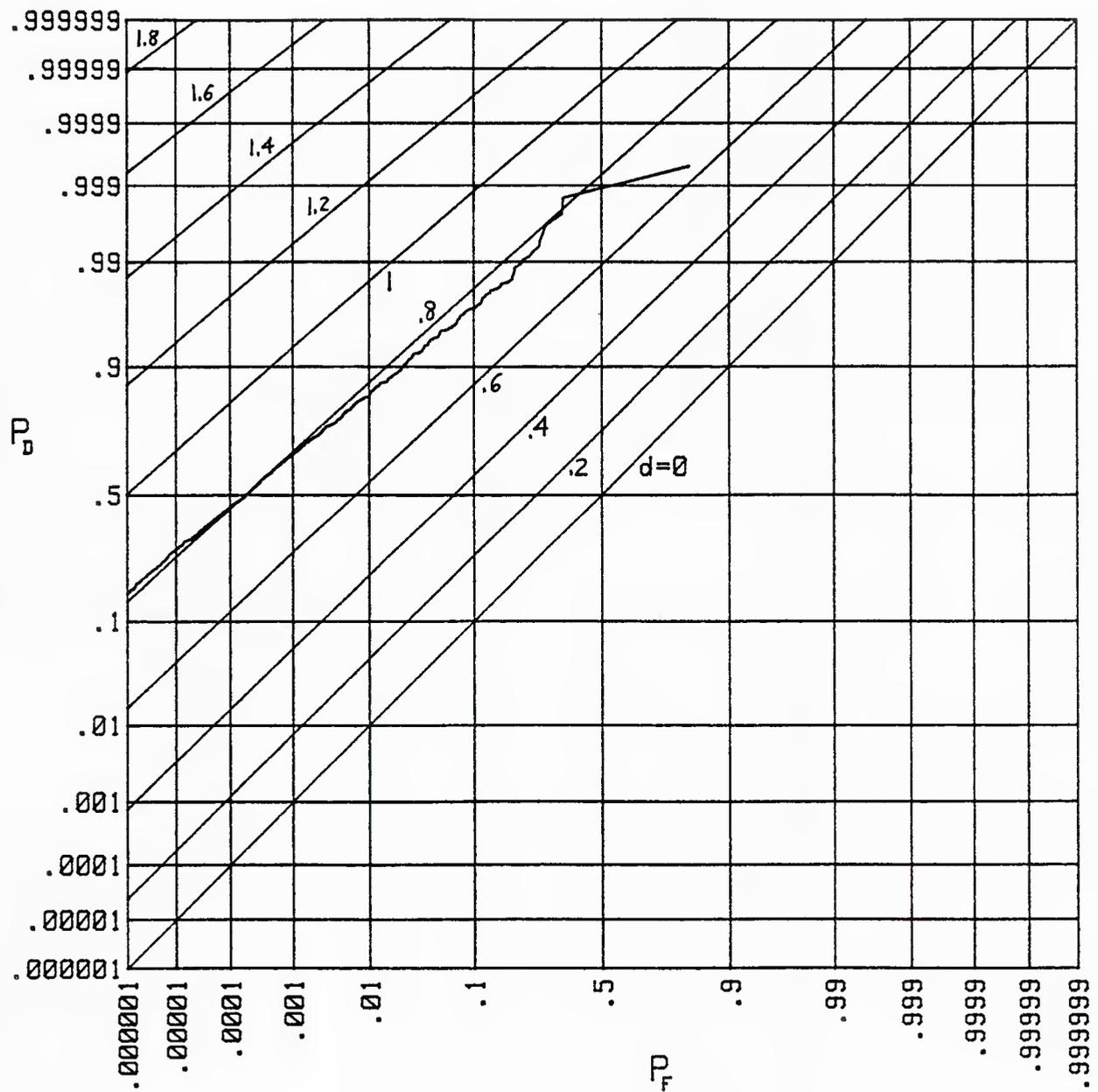
Figure 6. Operating Characteristics for N=2, K=2, M=32
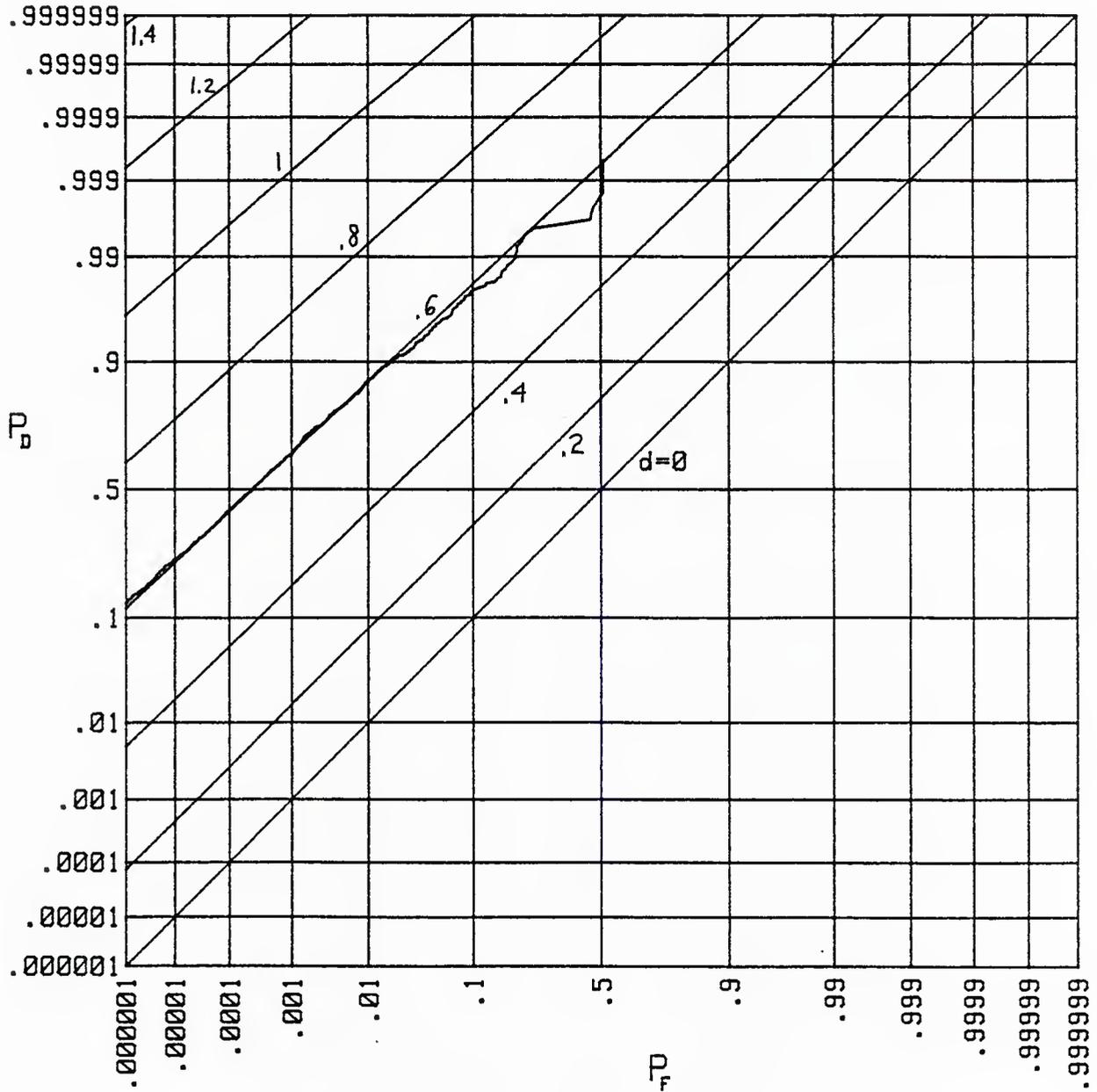
Figure 7.   Operating Characteristics for N=2, K=2, M=64
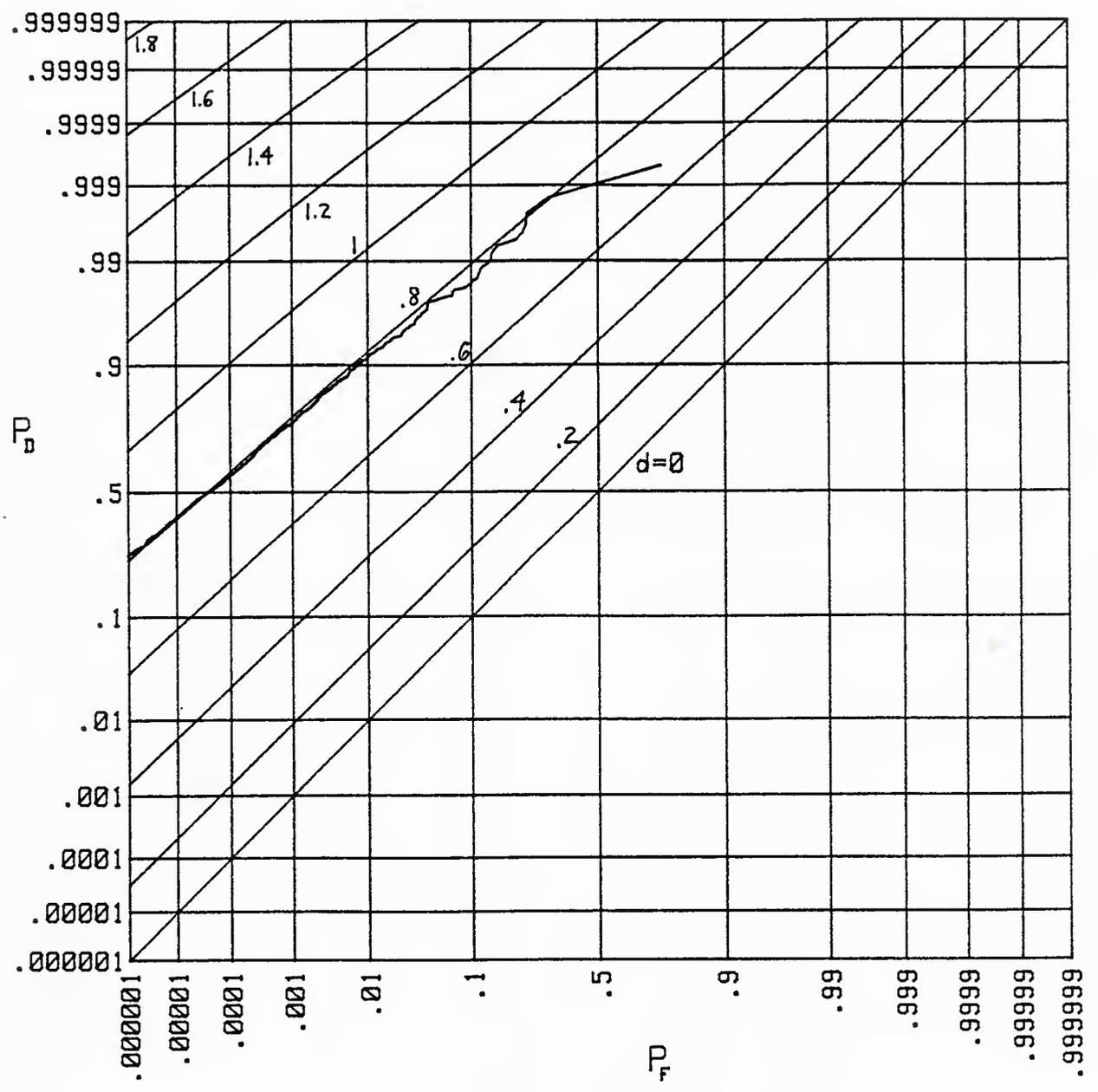
Figure 8.   Operating Characteristics for N=2, K=4, M=32
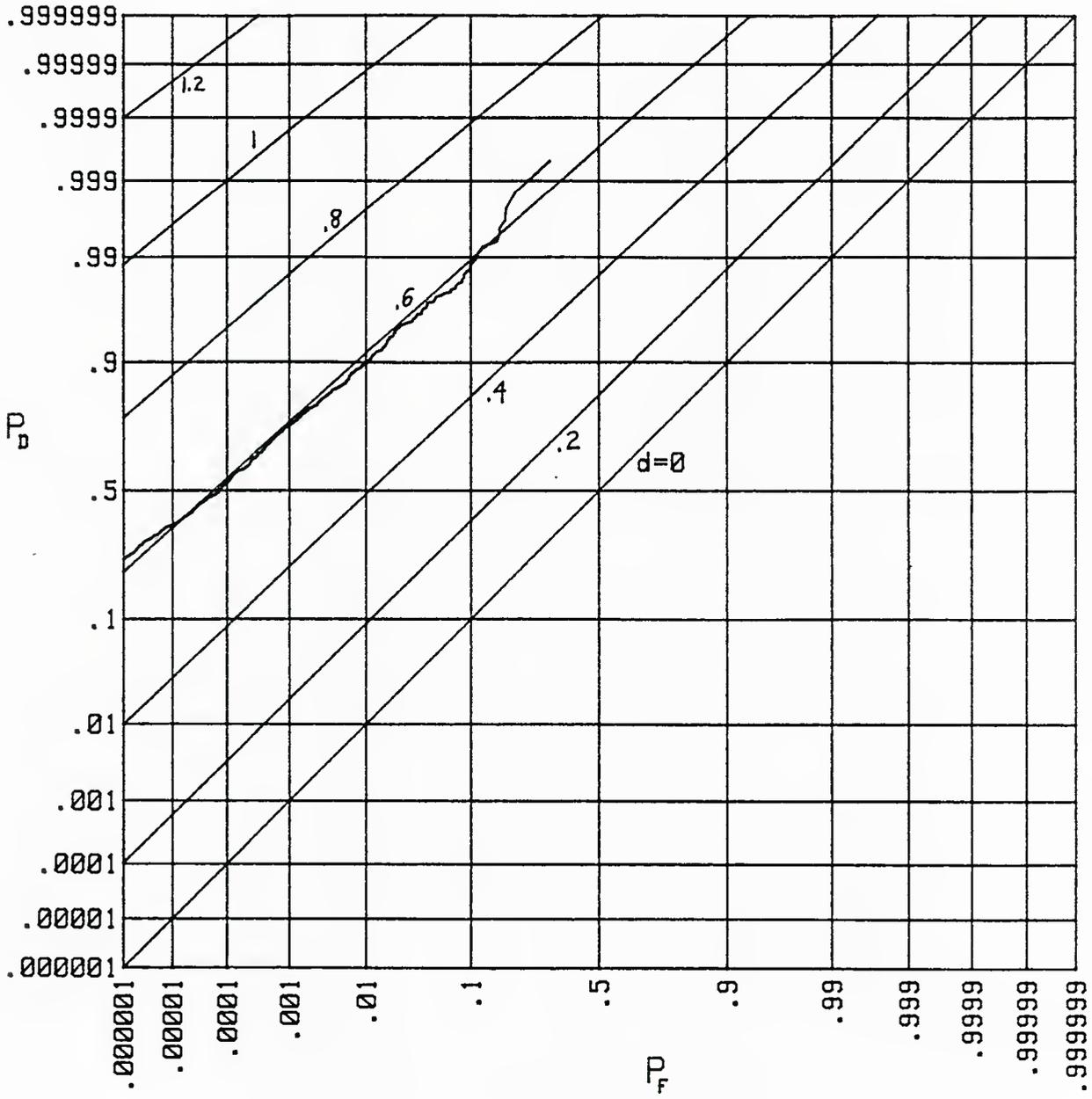
Figure 9. Operating Characteristics for N=2, K=4, M=64
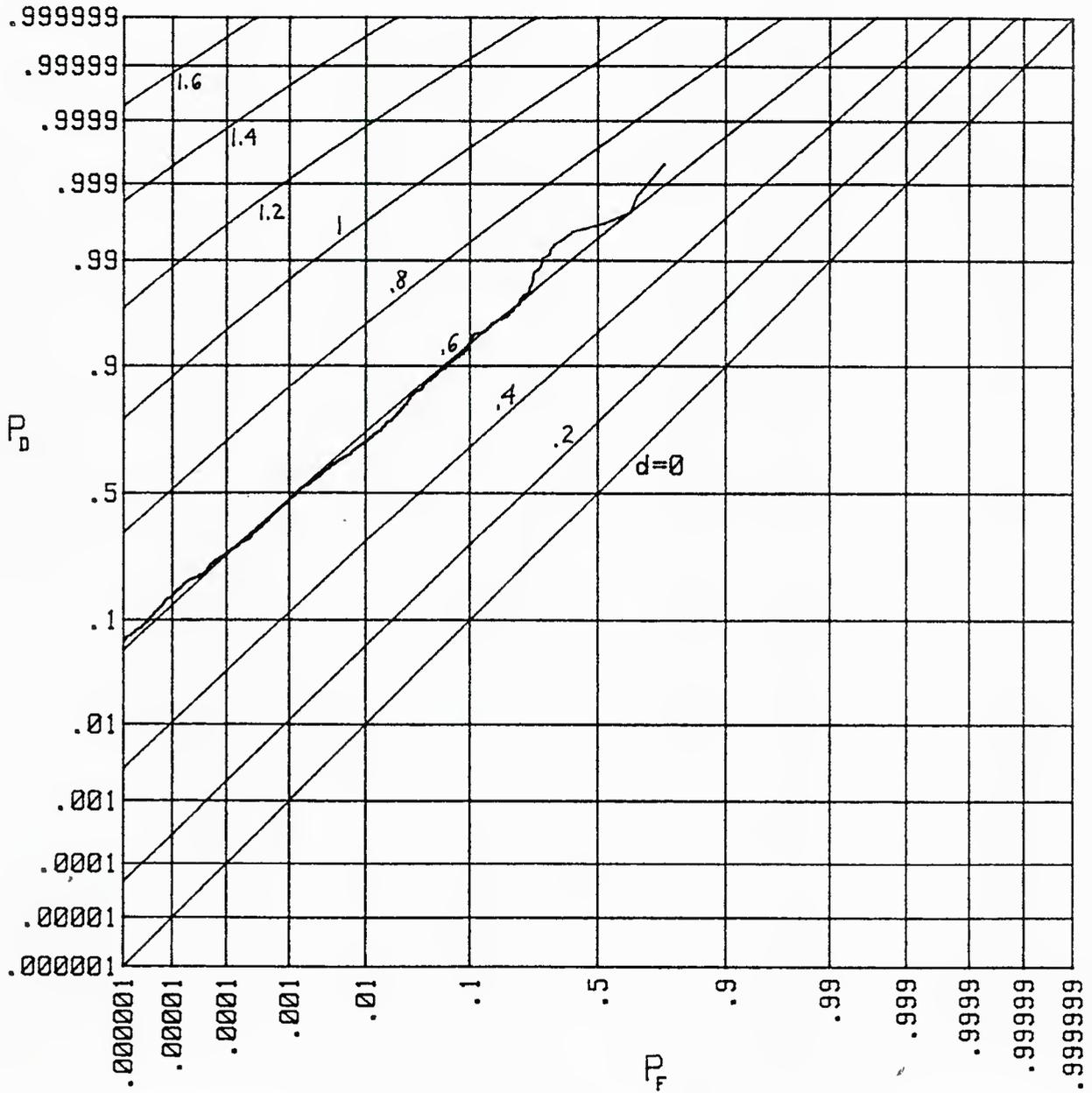
Figure 10. Operating Characteristics for N=2, K=8, M=32
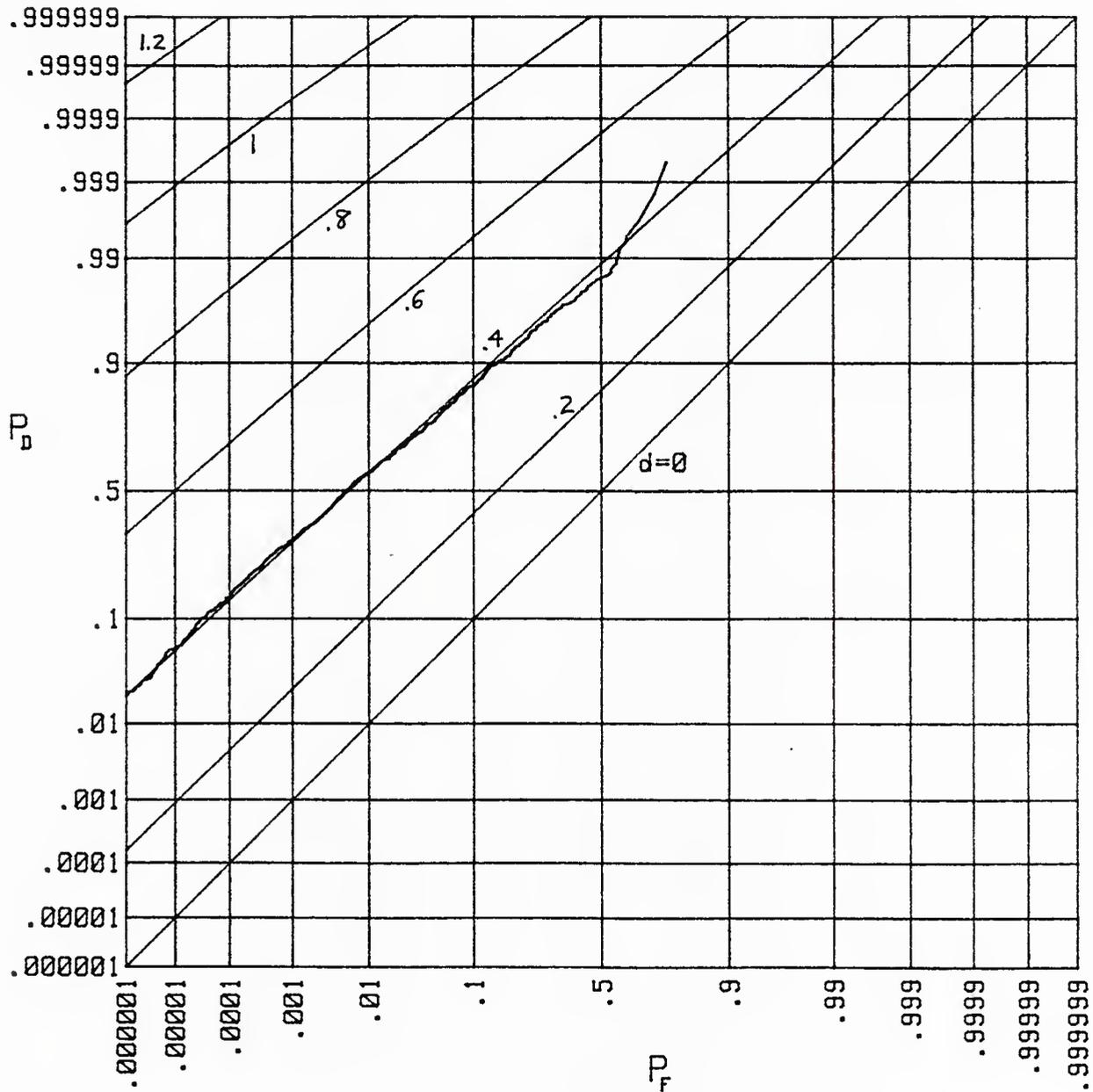
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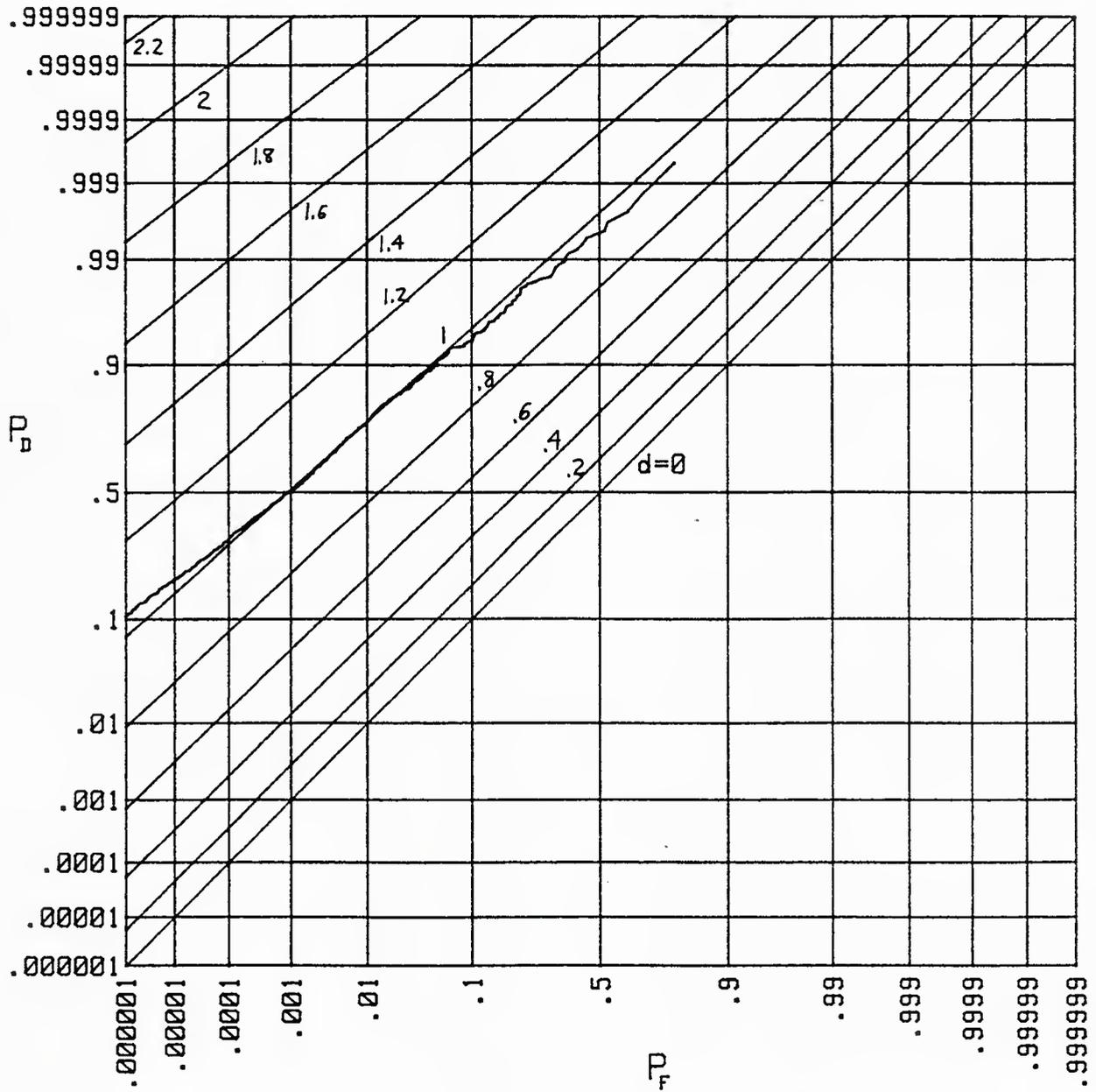
Figure 12. Operating Characteristics for N=4, K=1, M=32
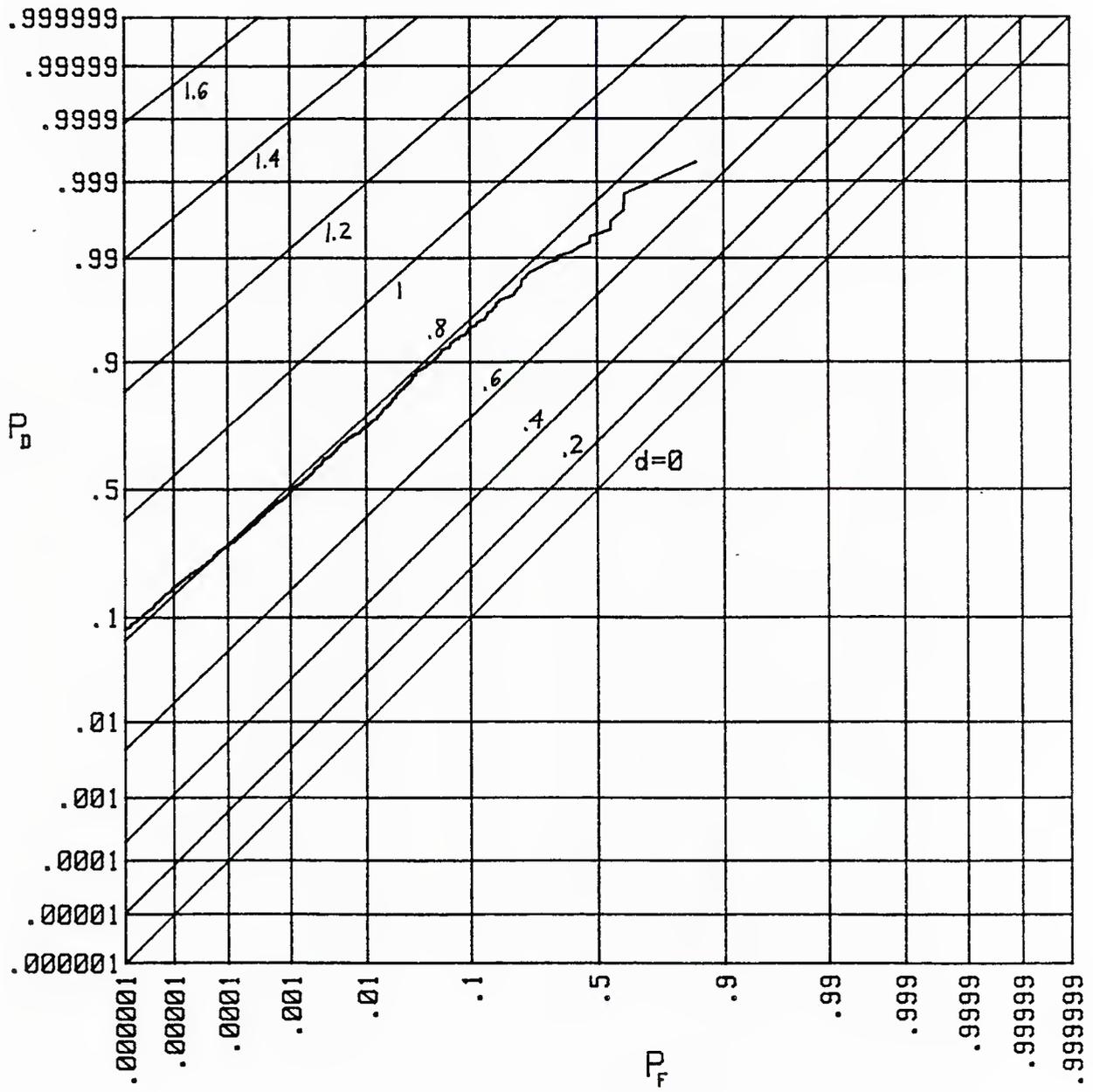
Figure 13. Operating Characteristics for N=4, K=1, M=64
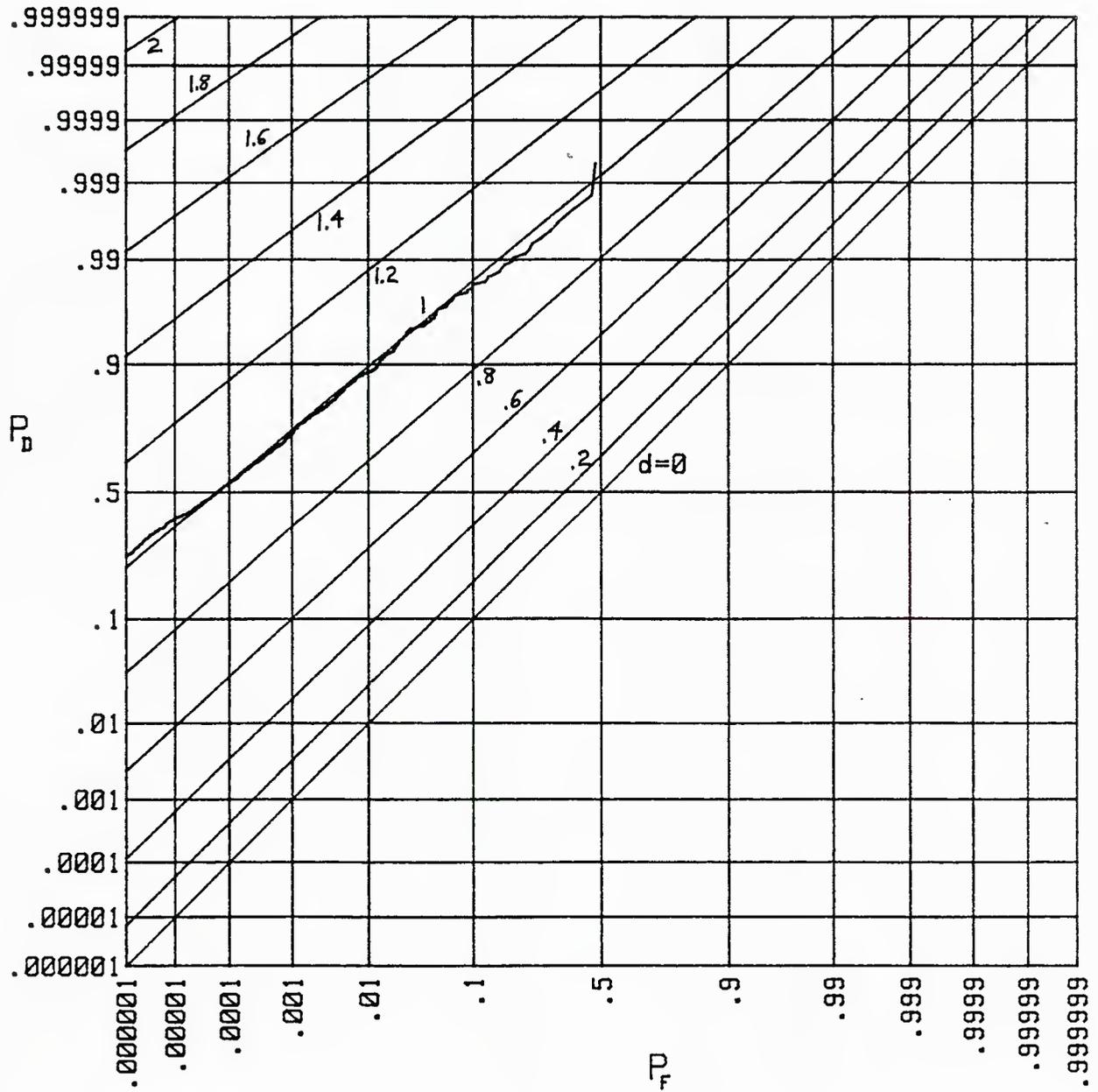
Figure 14. Operating Characteristics for N=4, K=2, M=32
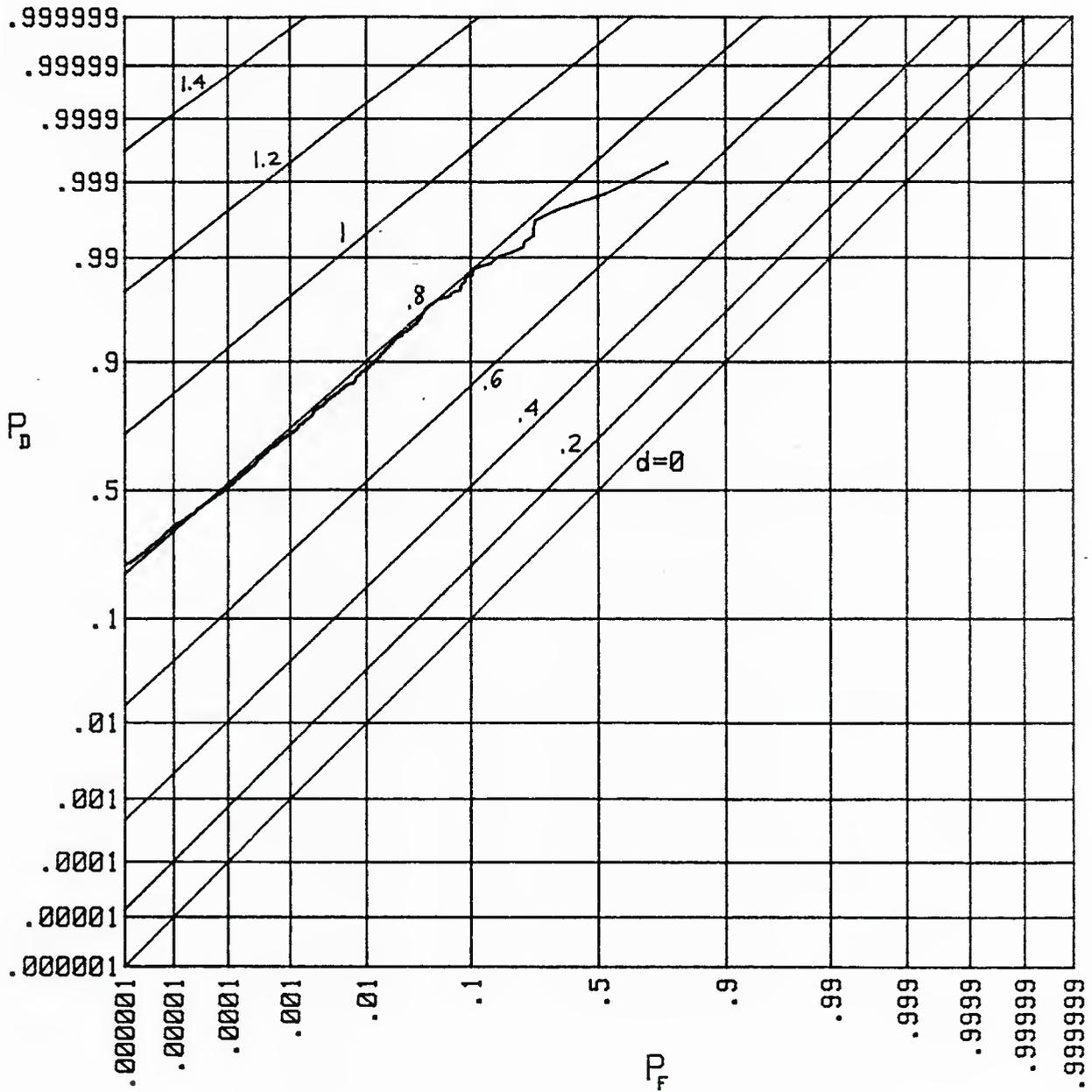
Figure 15. Operating Characteristics for N=4, K=2, M=64
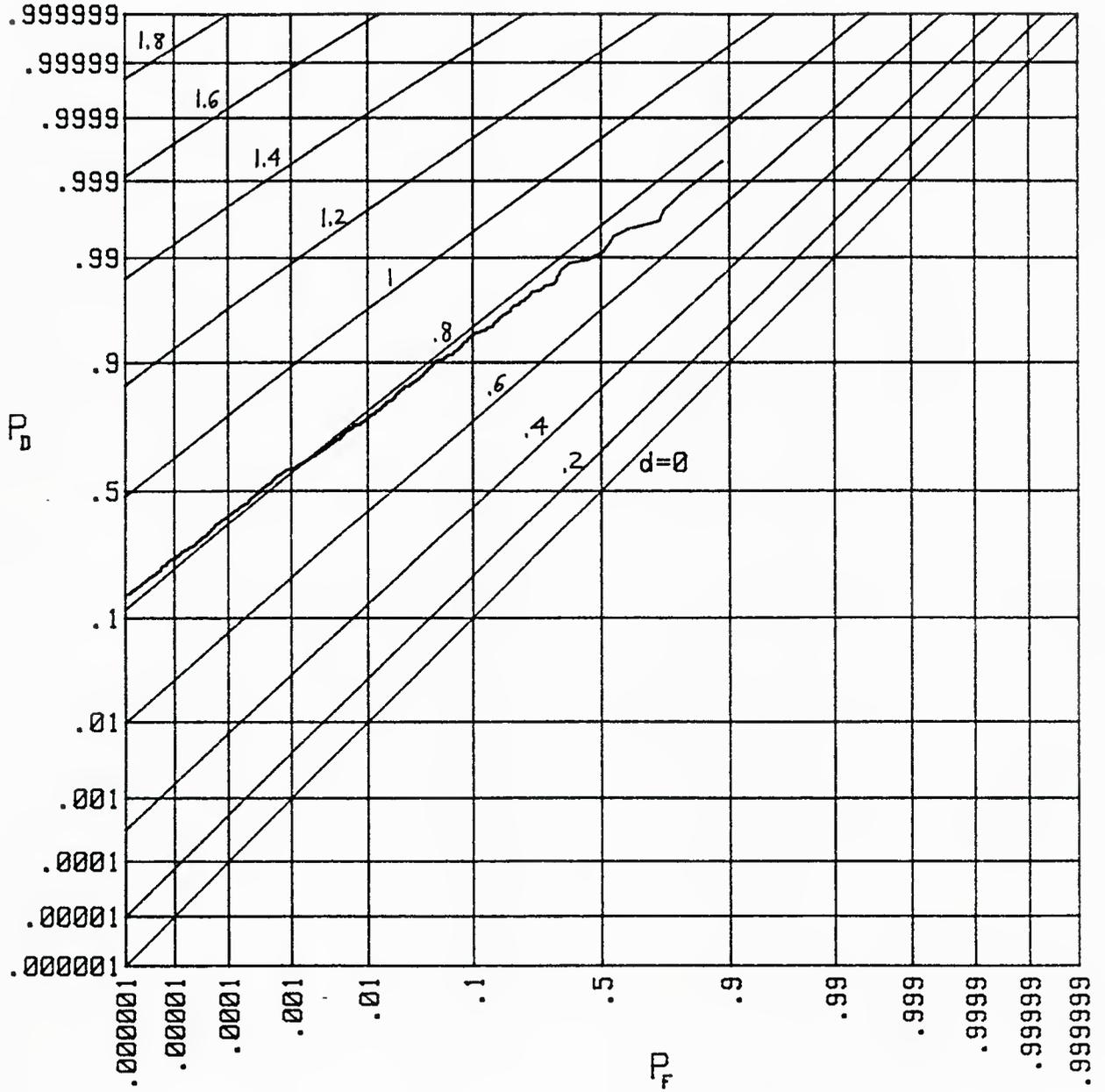
Figure 16.   Operating Characteristics for N=4, K=4, M=32
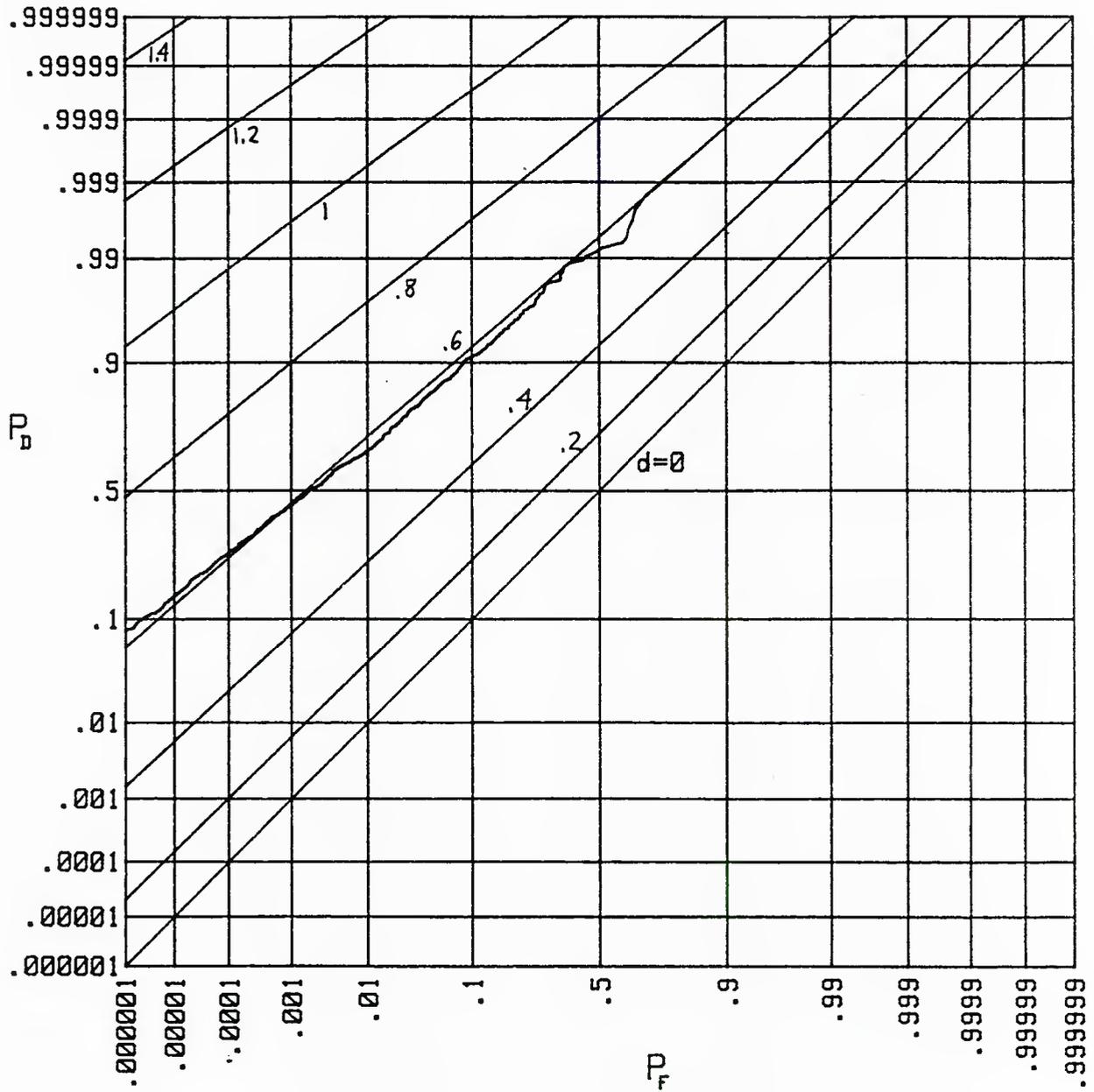
Figure 17. Operating Characteristics for N=4, K=4, M=64
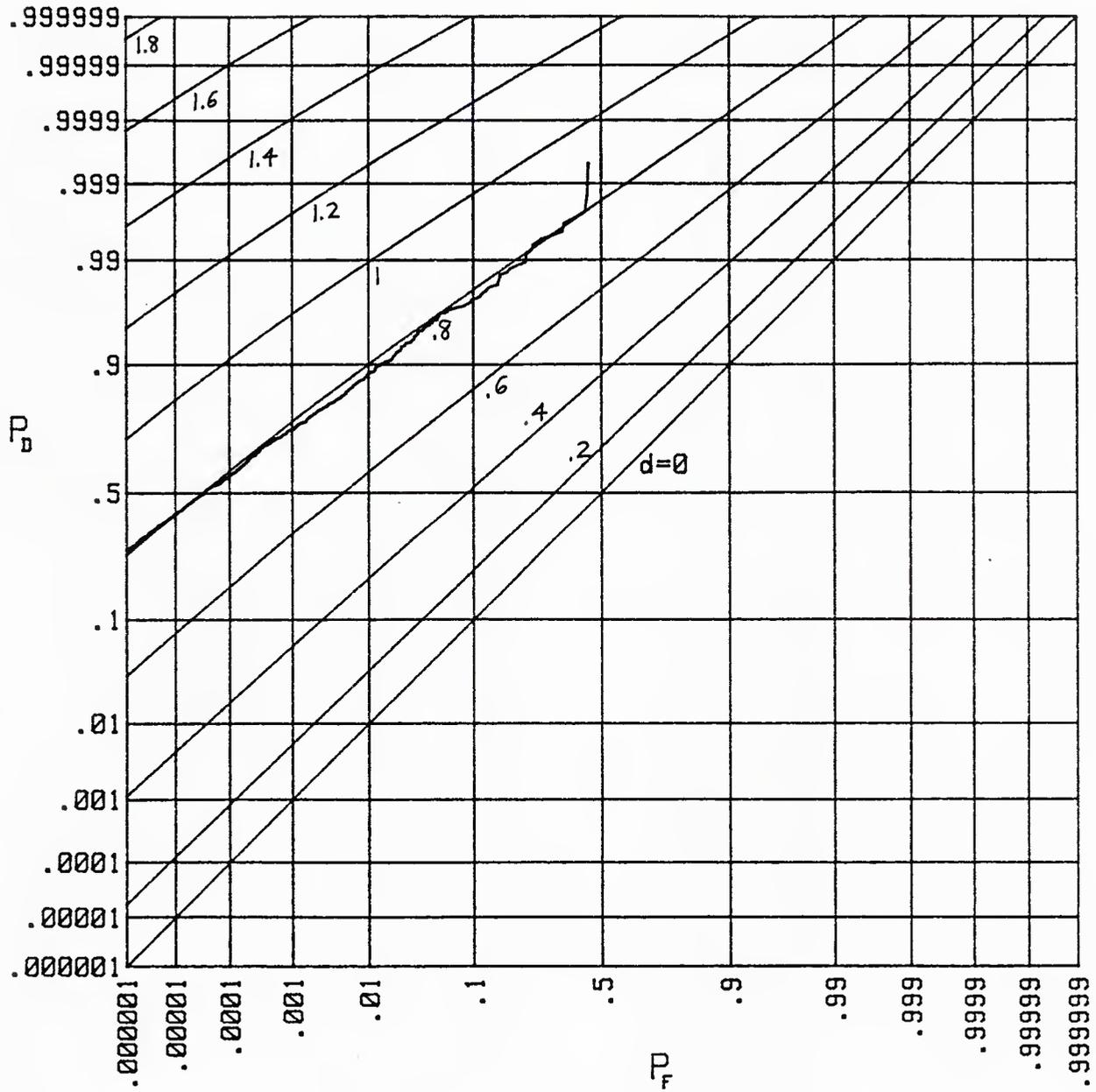
Figure 18.   Operating Characteristics for N=4, K=8, M=32
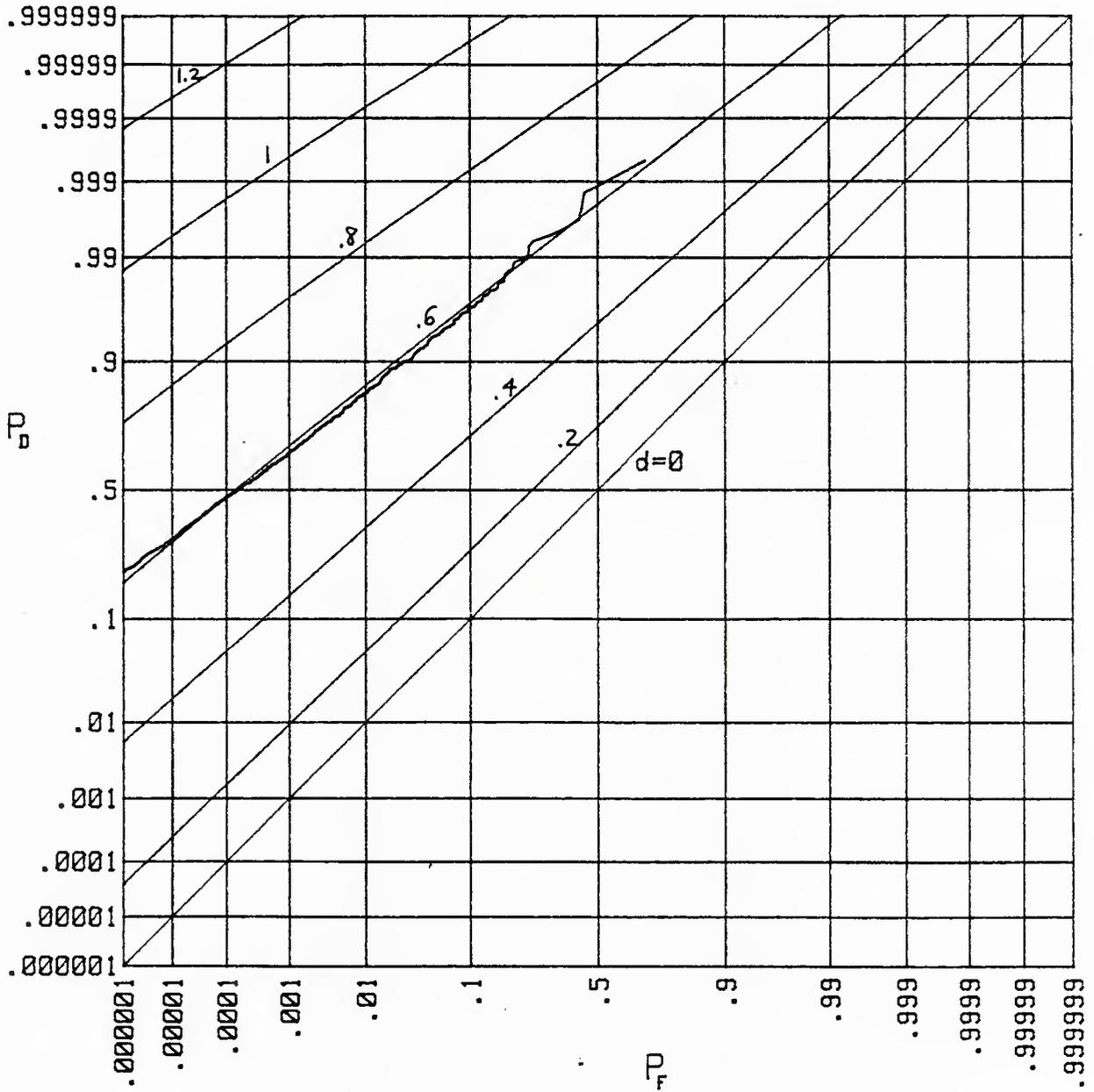
Figure 19. Operating Characteristics for N=4, K=8, M=64
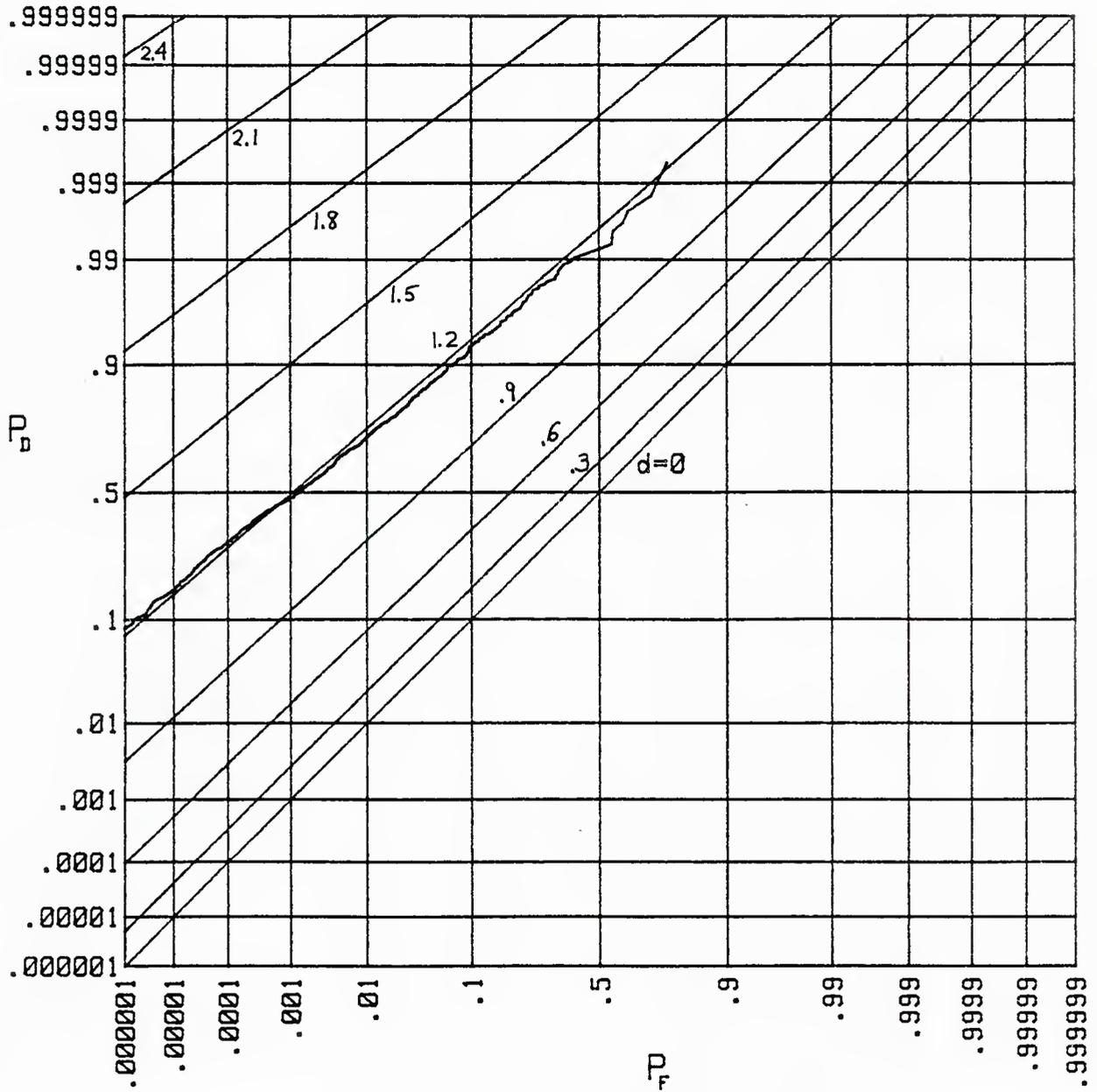
Figure 20.   Operating Characteristics for N=8, K=1, M=32
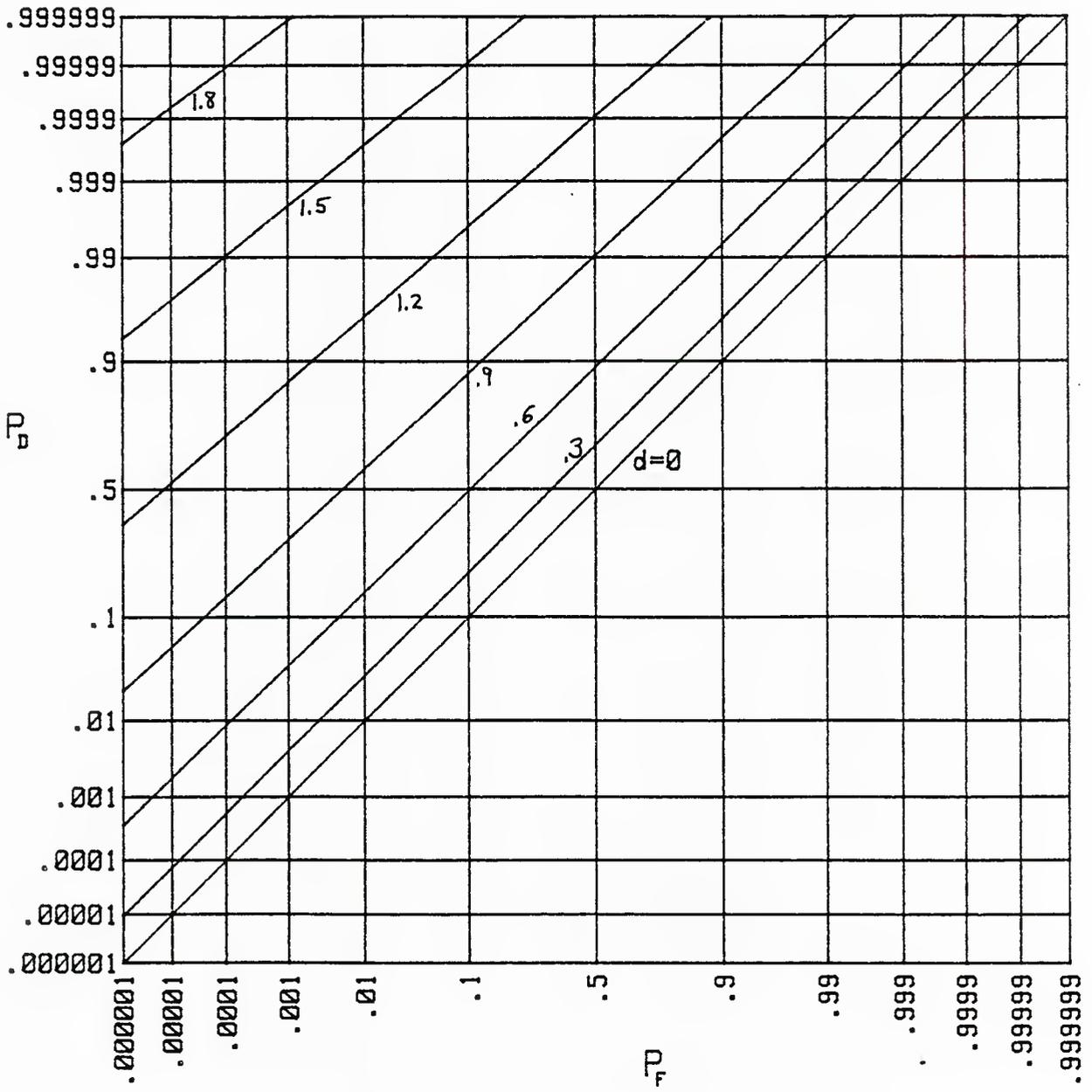
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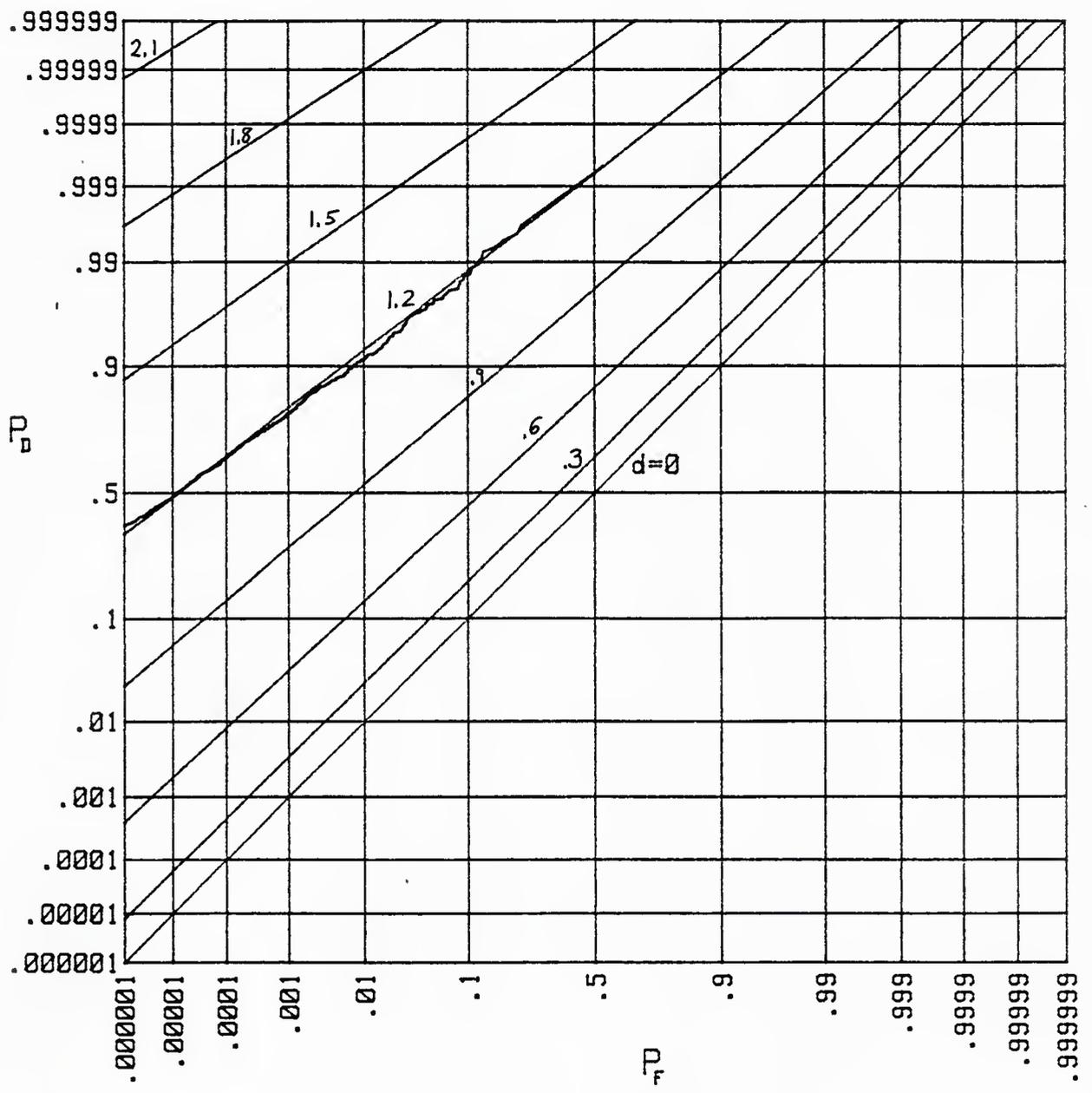
Figure 22.   Operating Characteristics for N=8, K=2, M=32
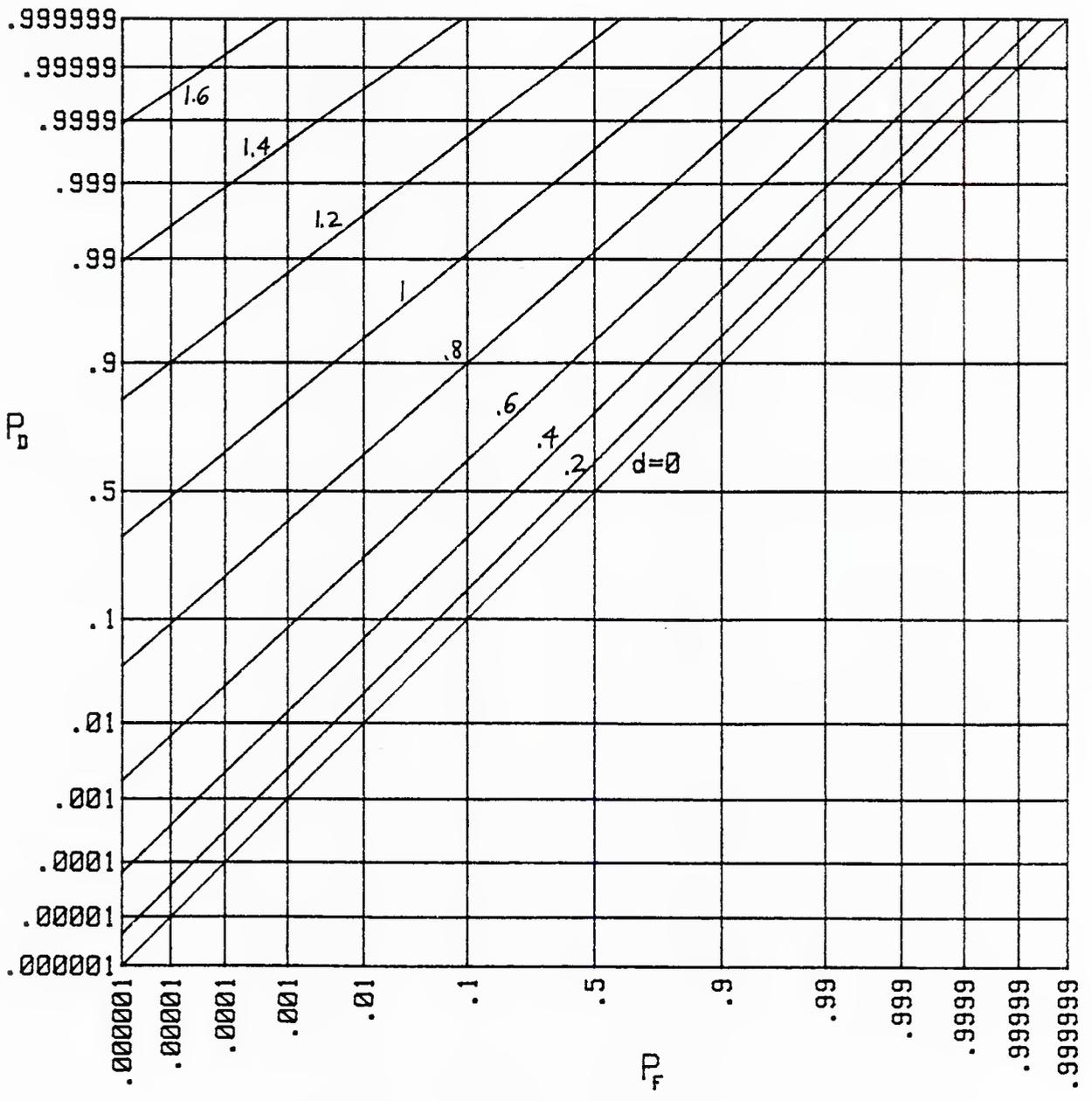
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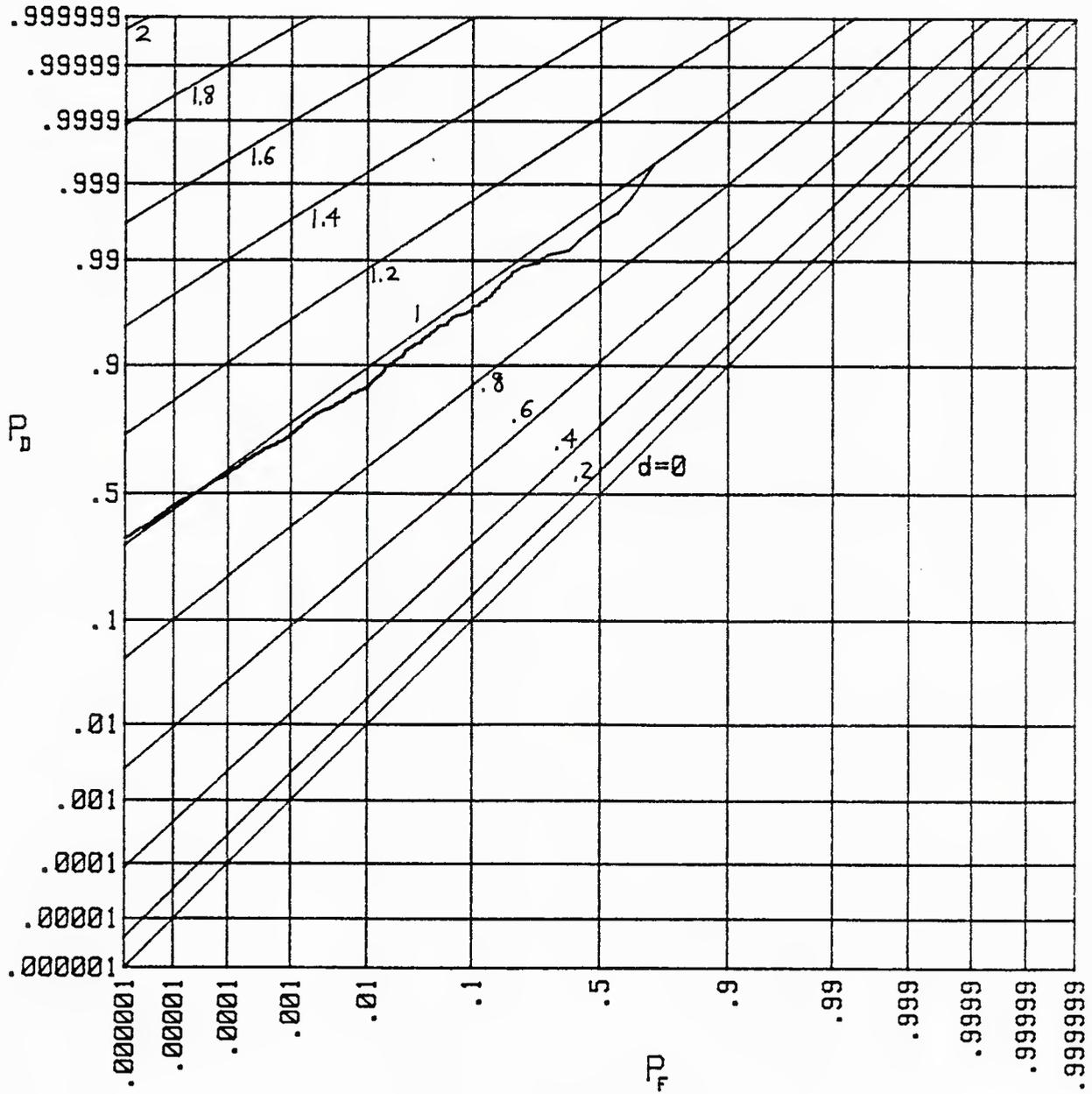
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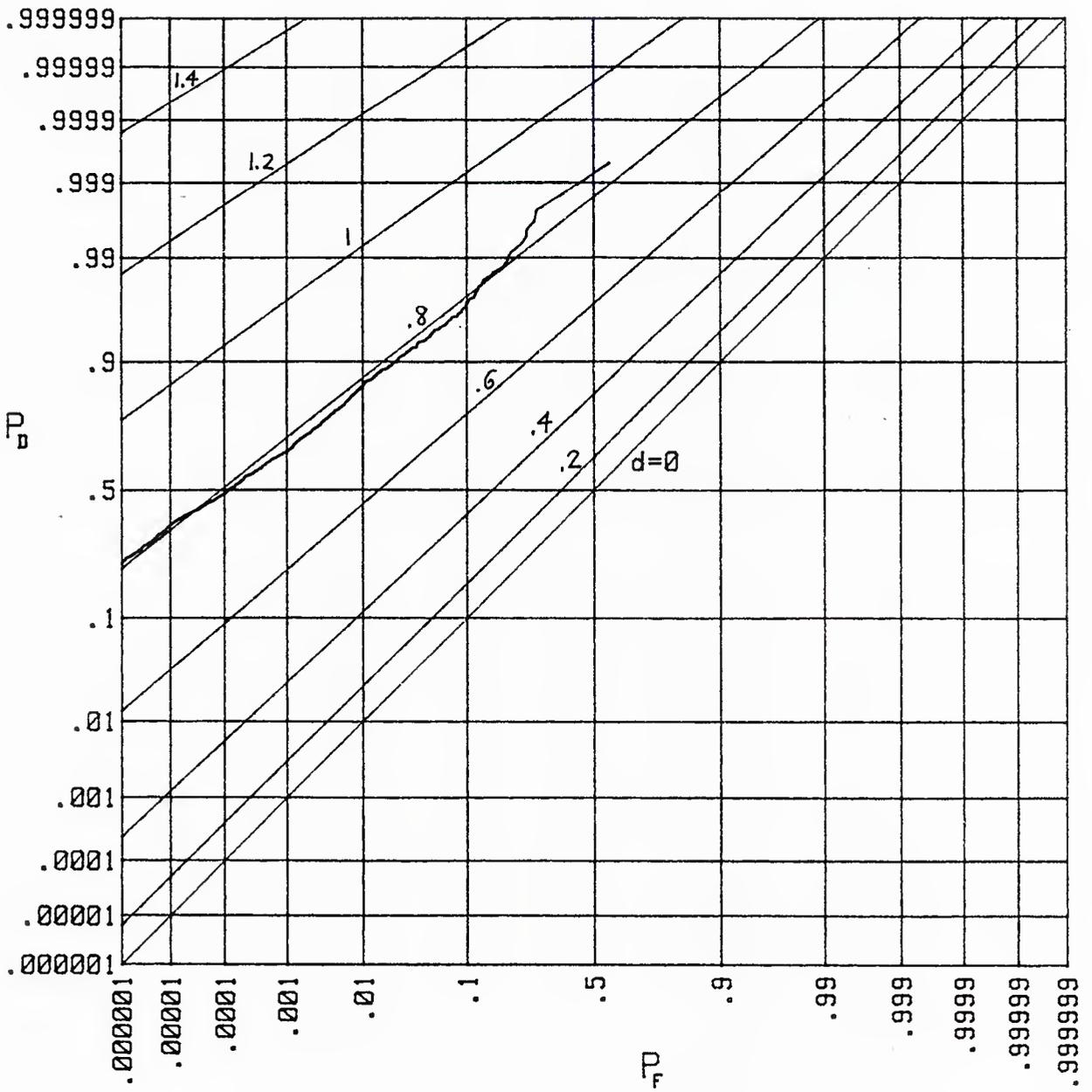
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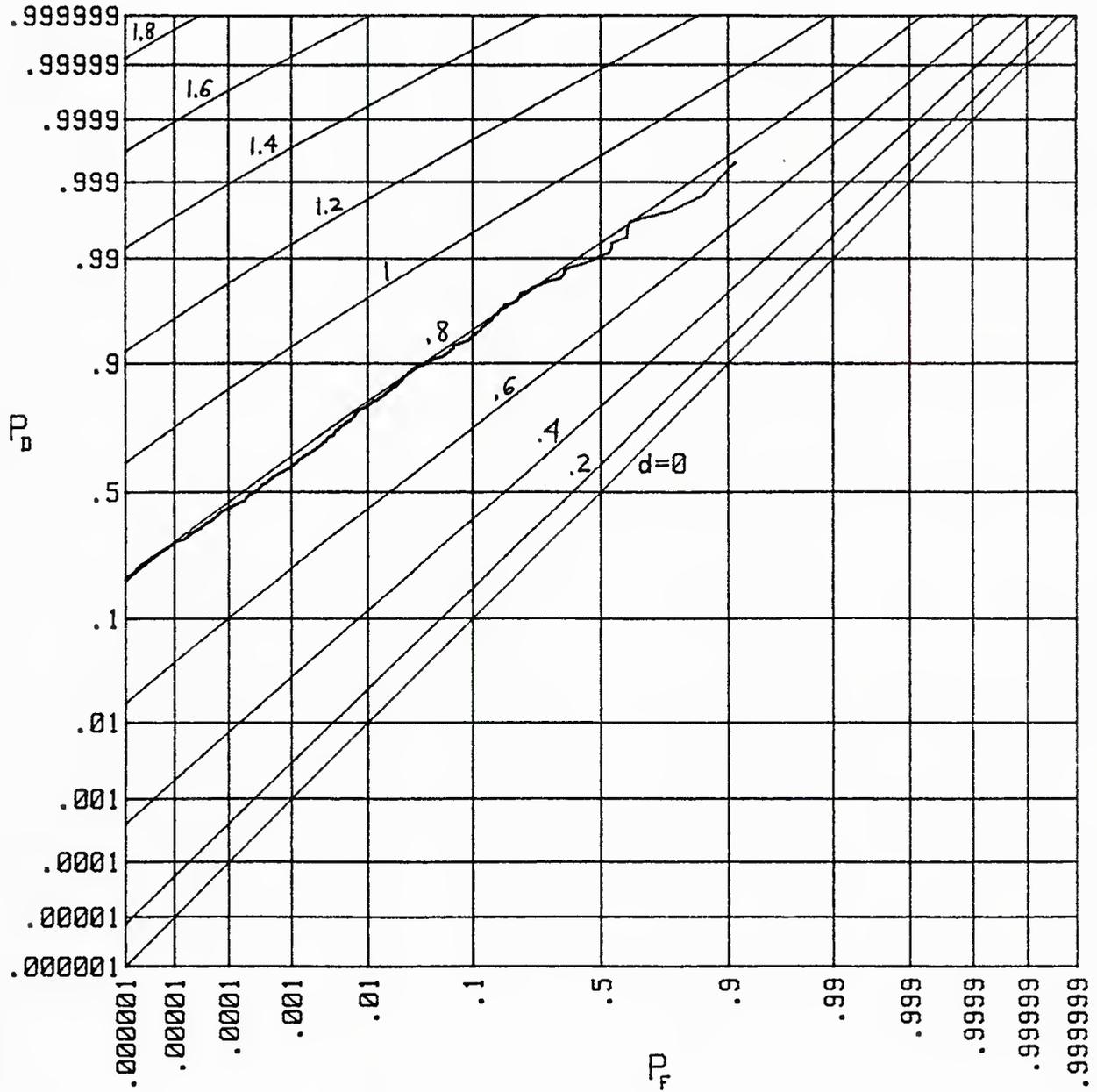
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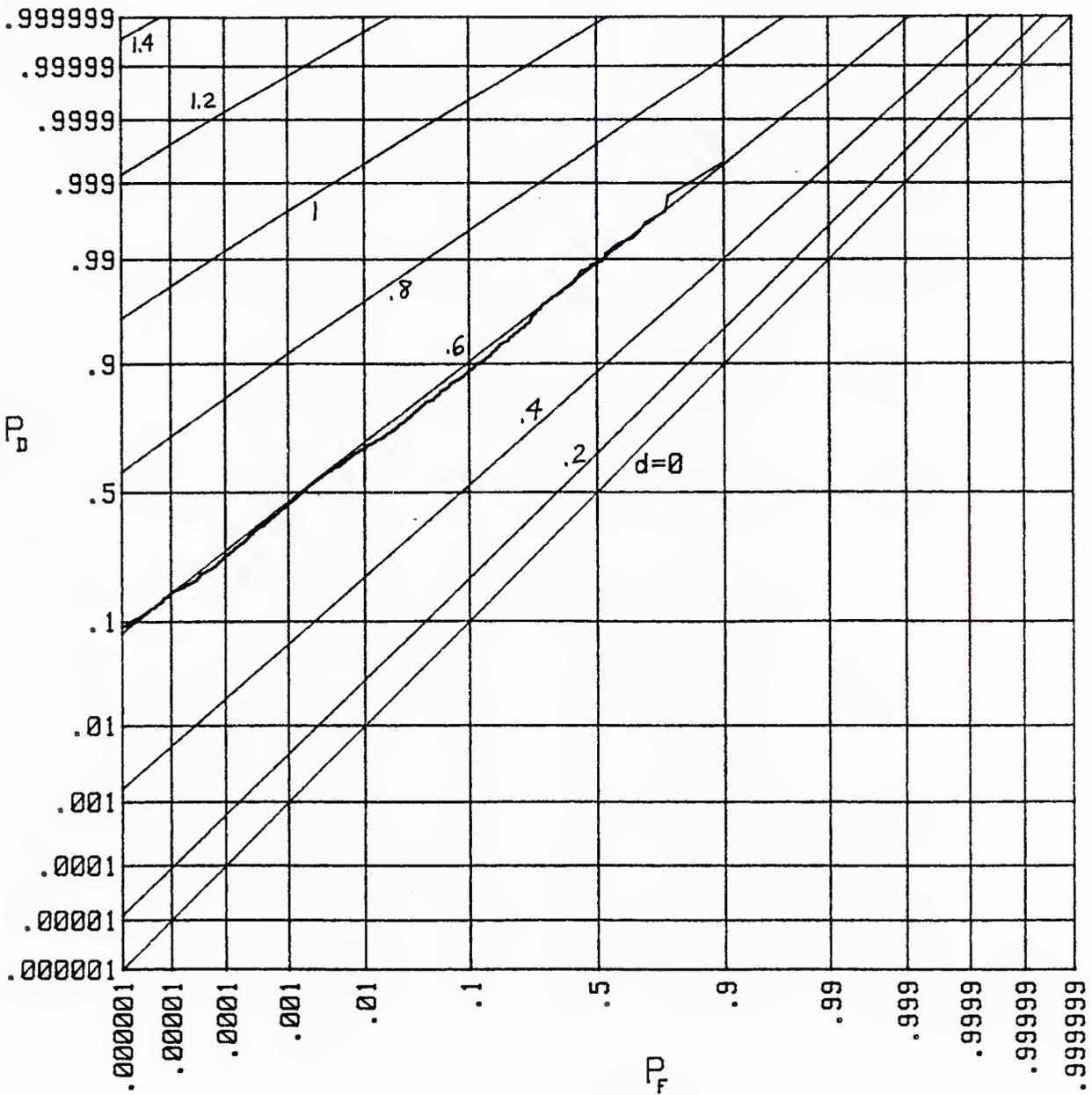
Figure 27.   Operating Characteristics for N=8, K=8, M=64
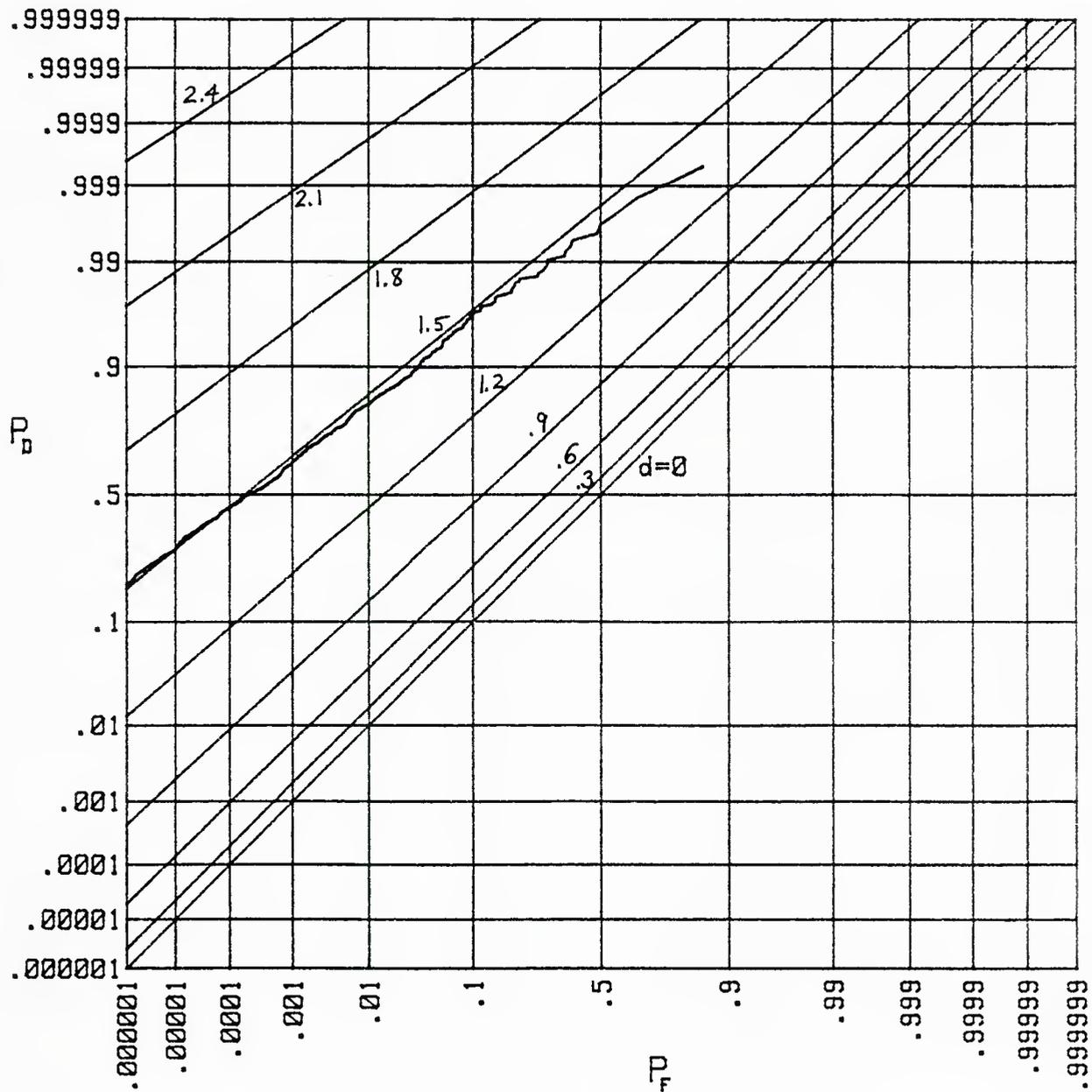
Figure 28. Operating Characteristics for N=16, K=1, M=32
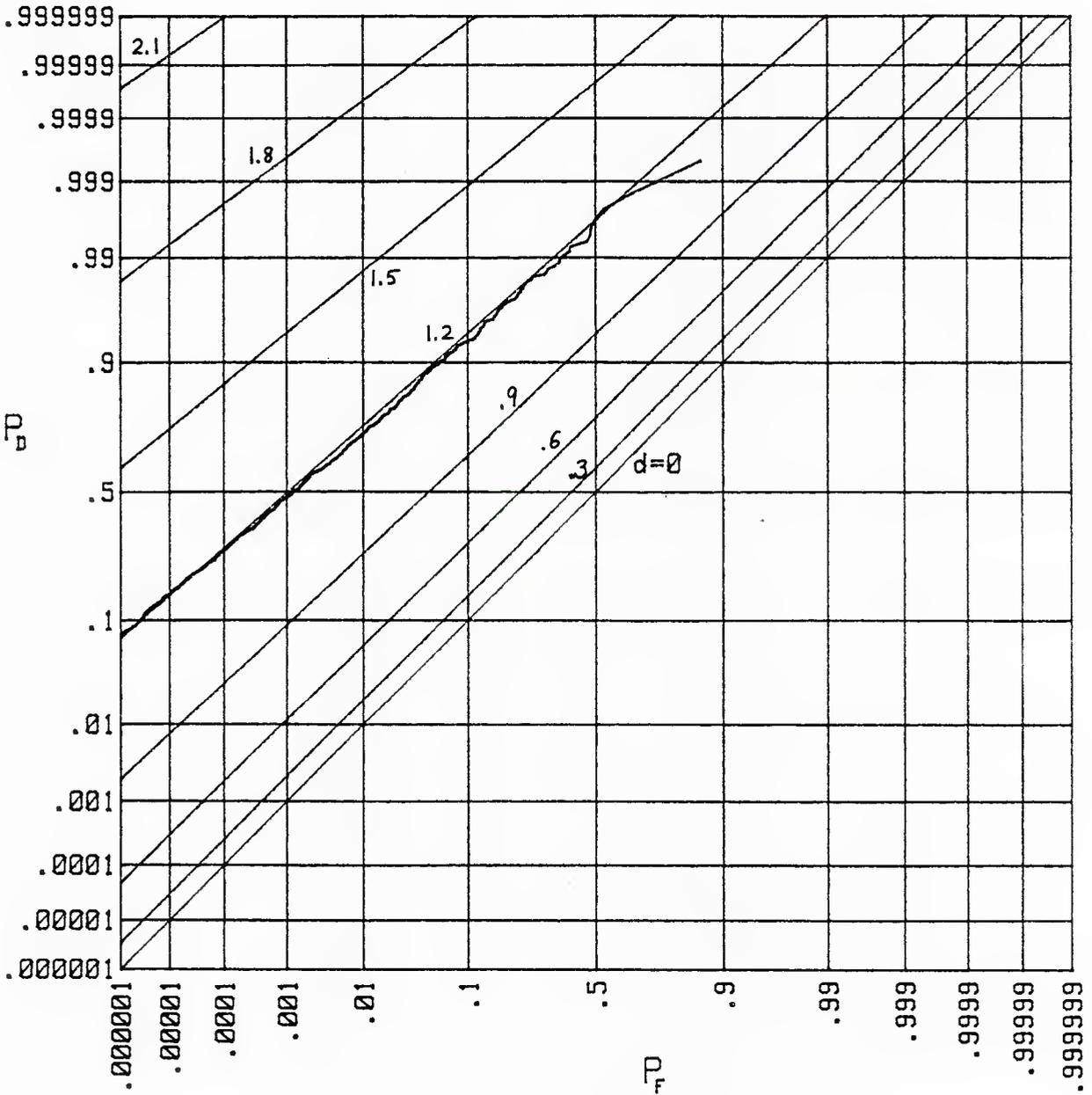
Figure 29. Operating Characteristics for N=16, K=1, M=64
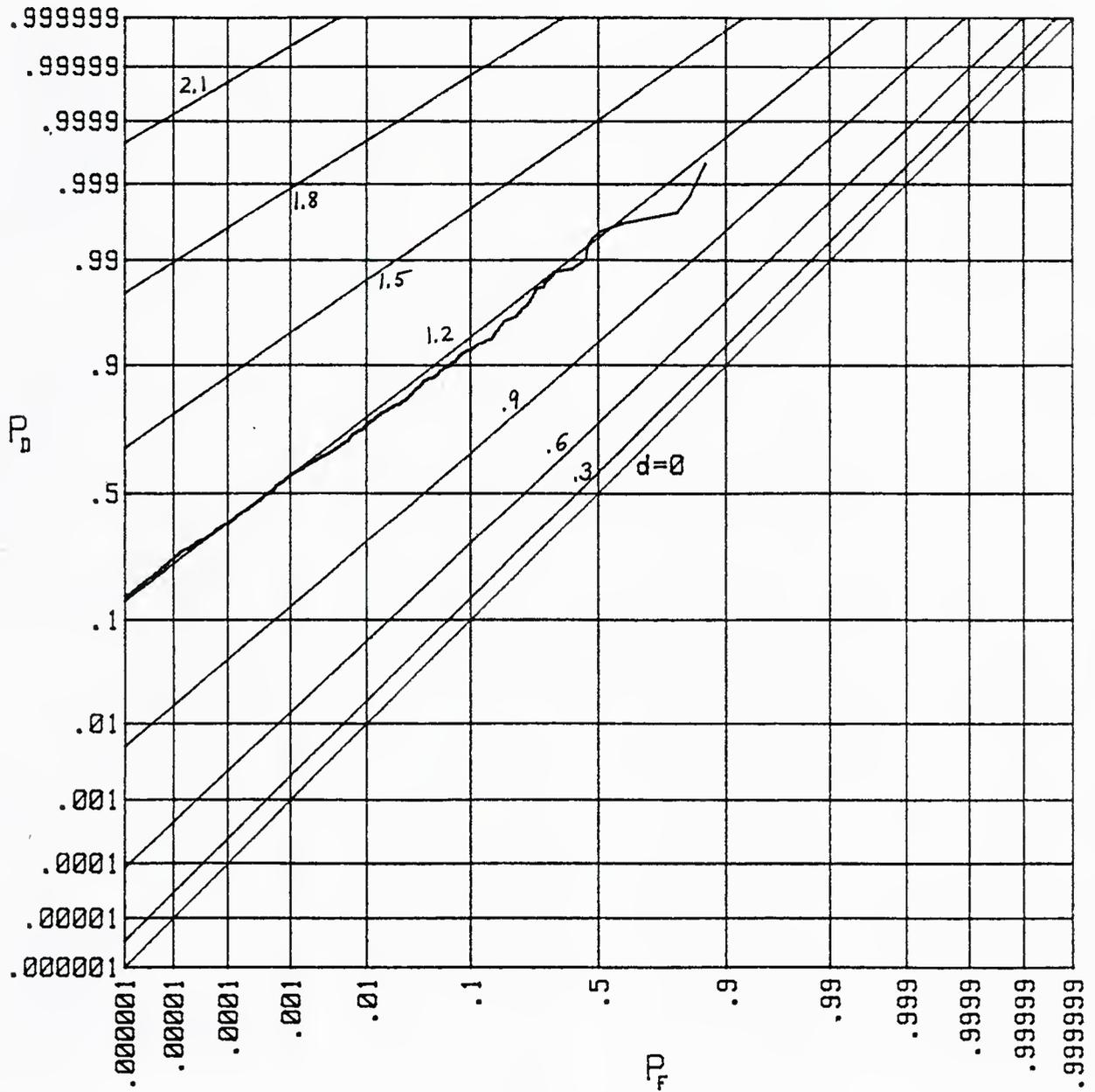
Figure 30.   Operating Characteristics for N=16, K=2, M=32
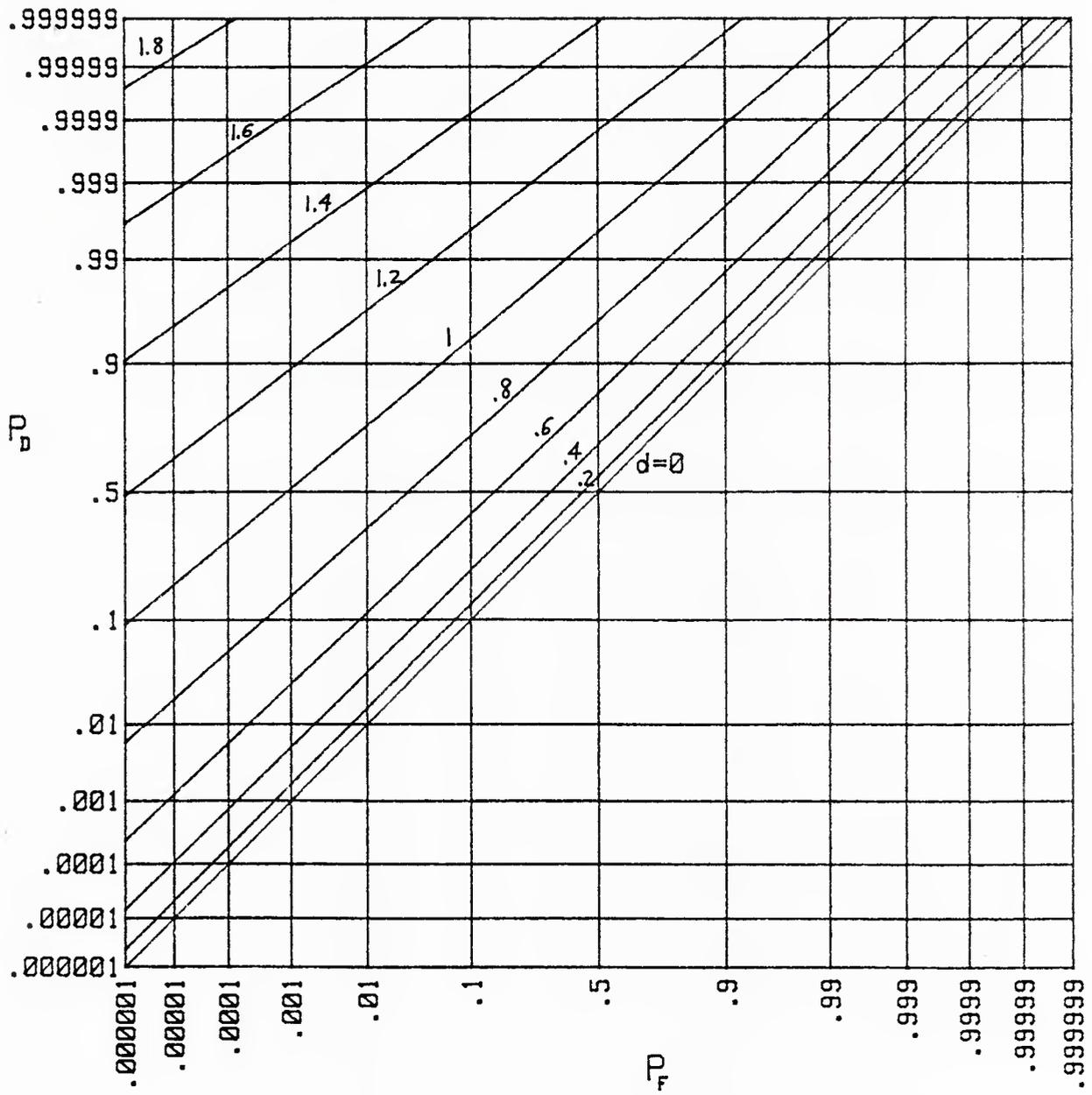
Figure 31.   Operating Characteristics for N=16, K=2, M=64
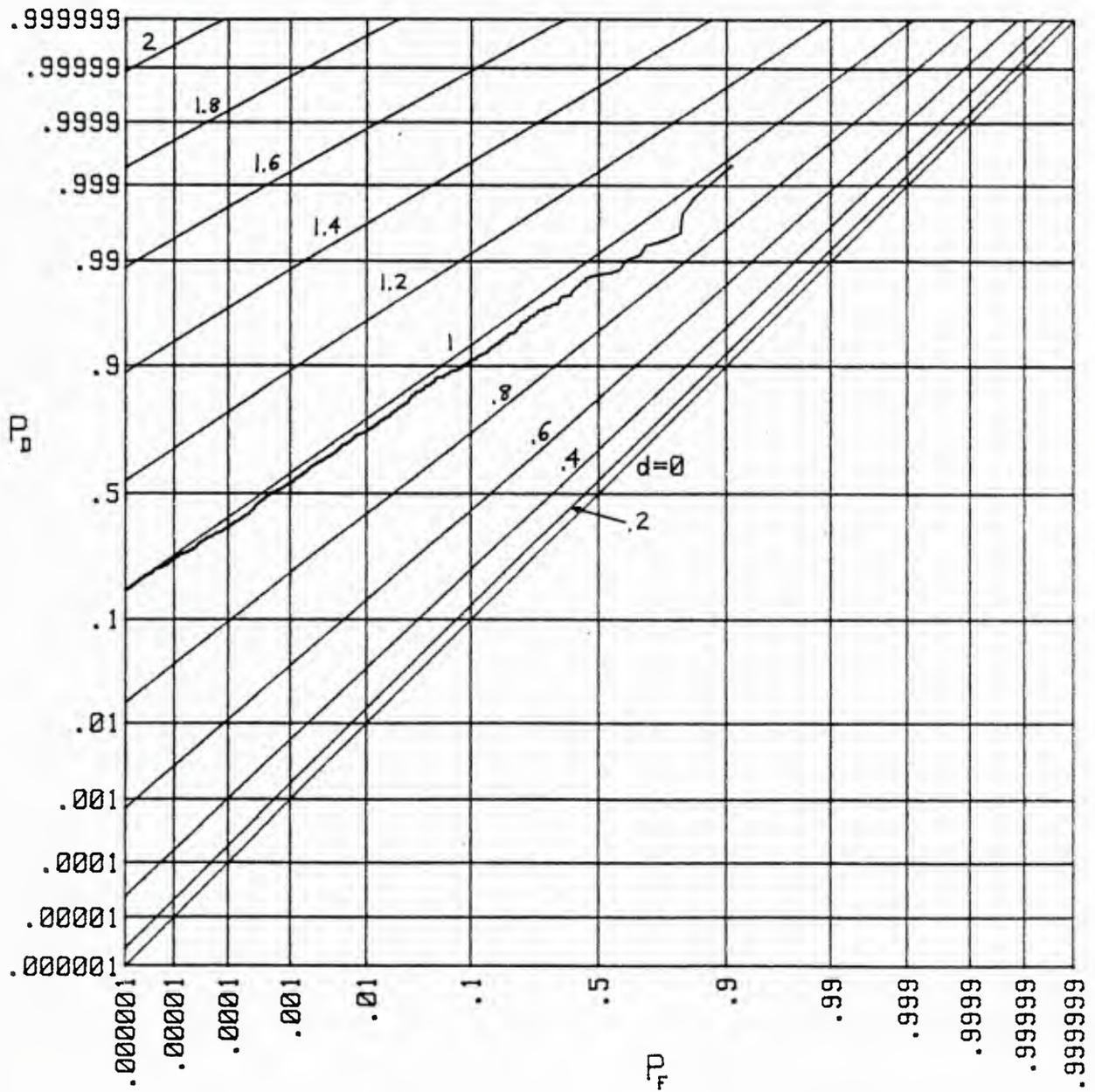
Figure 32.   Operating Characteristics for N=16, K=4, M=32
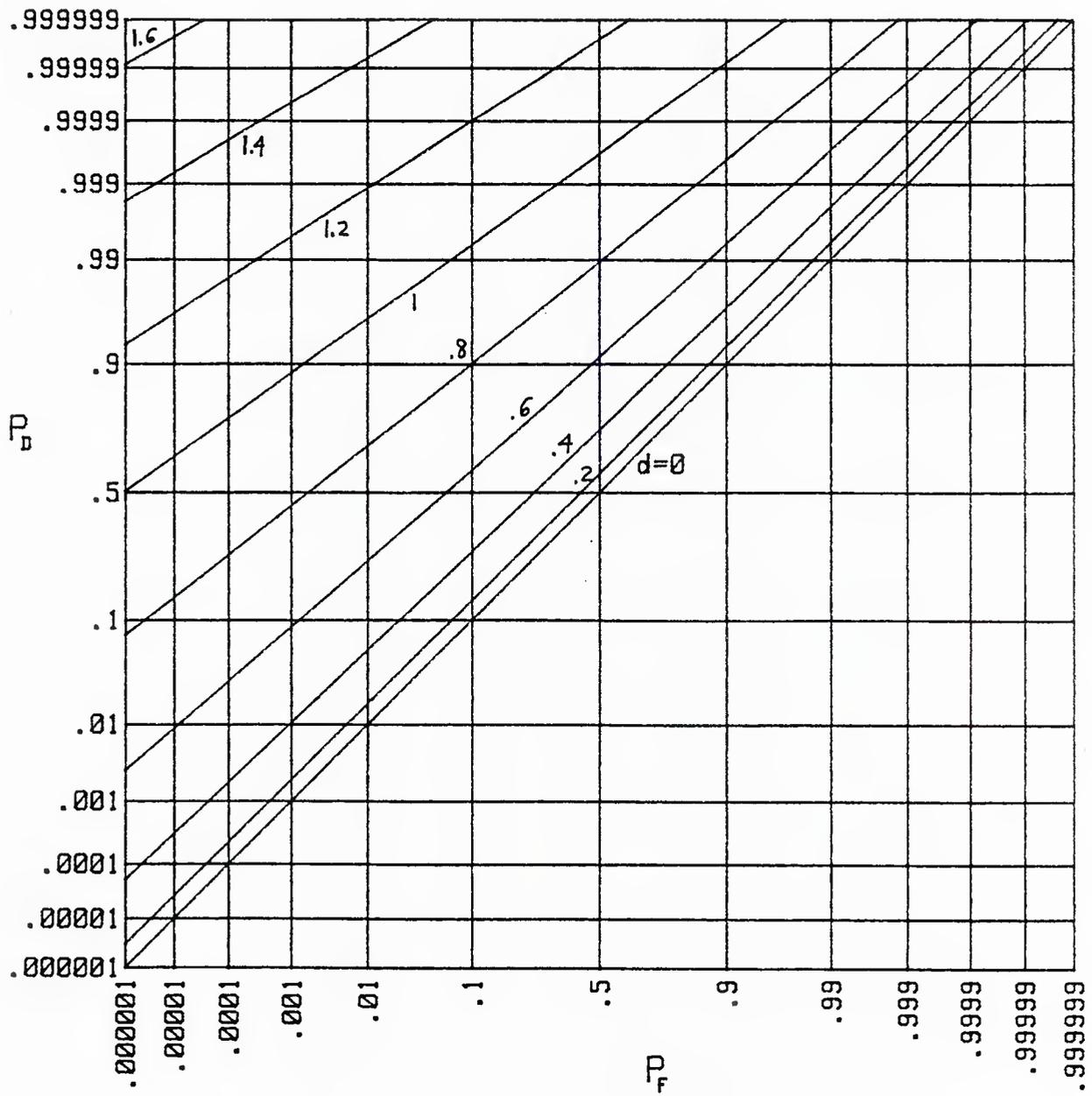
Figure 33.   Operating Characteristics for N=16, K=4, M=64
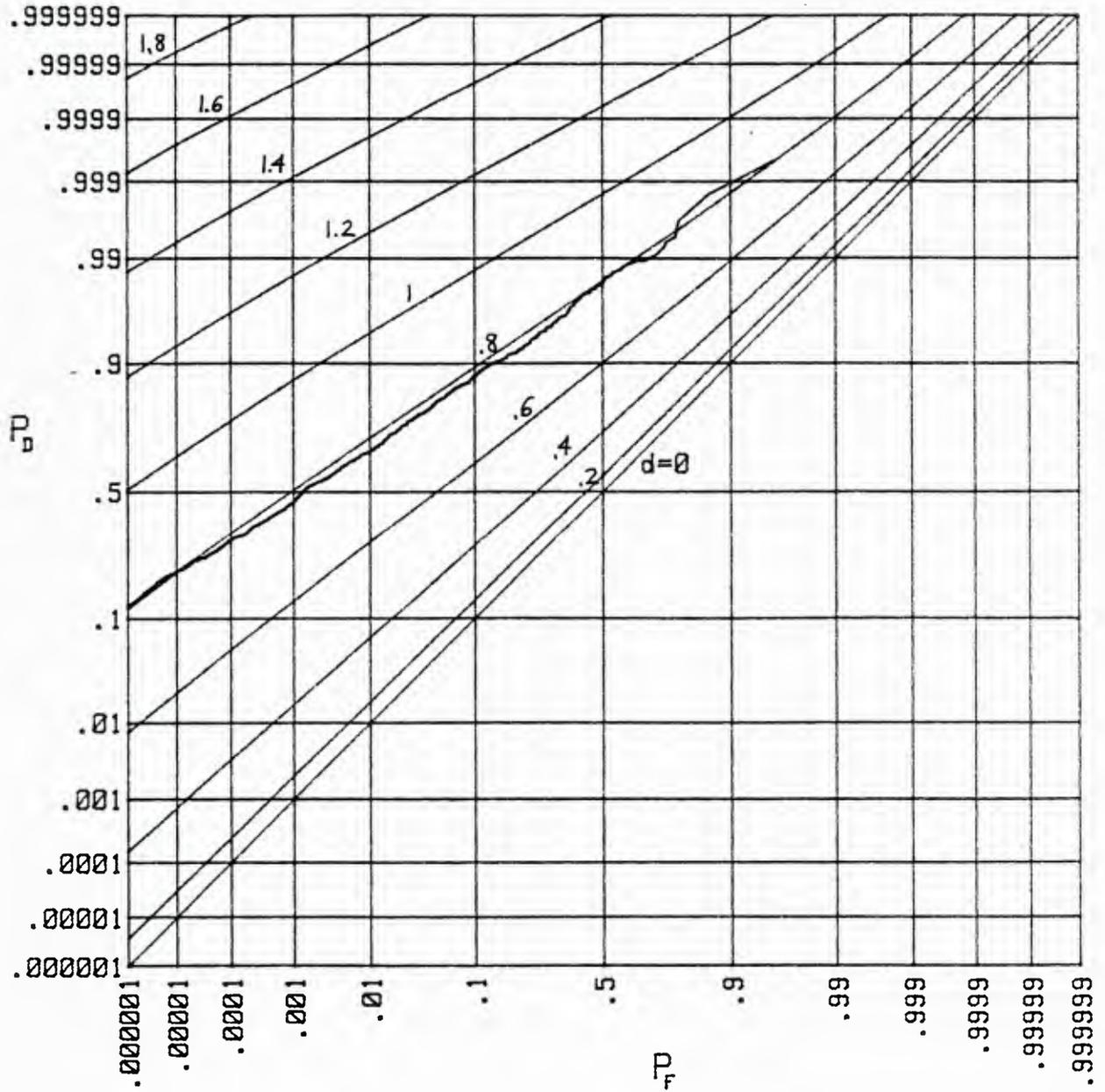
Figure 34.   Operating Characteristics for N=16, K=8, M=32
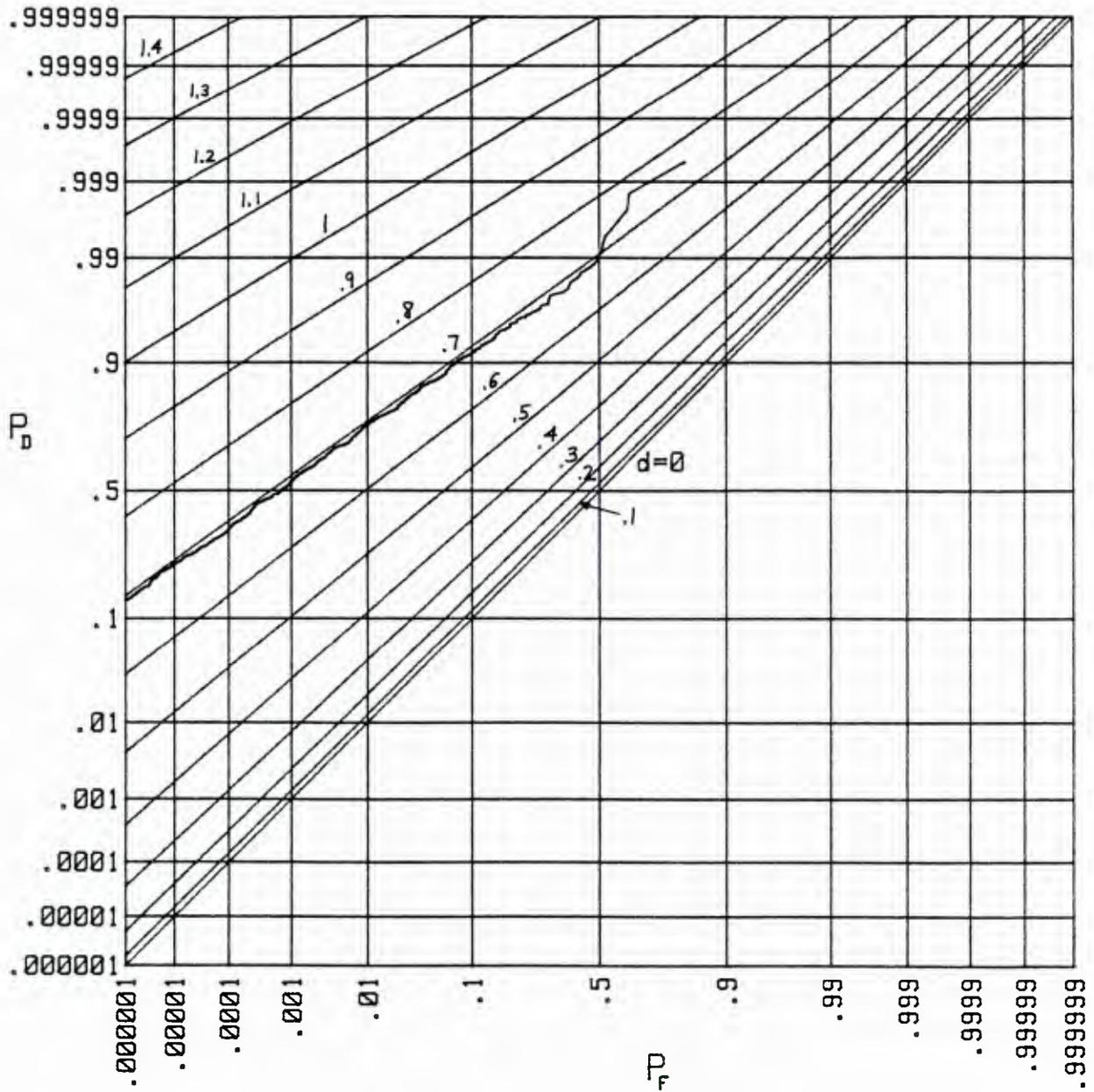
Figure 35. Operating Characteristics for N=16, K=8, M=64
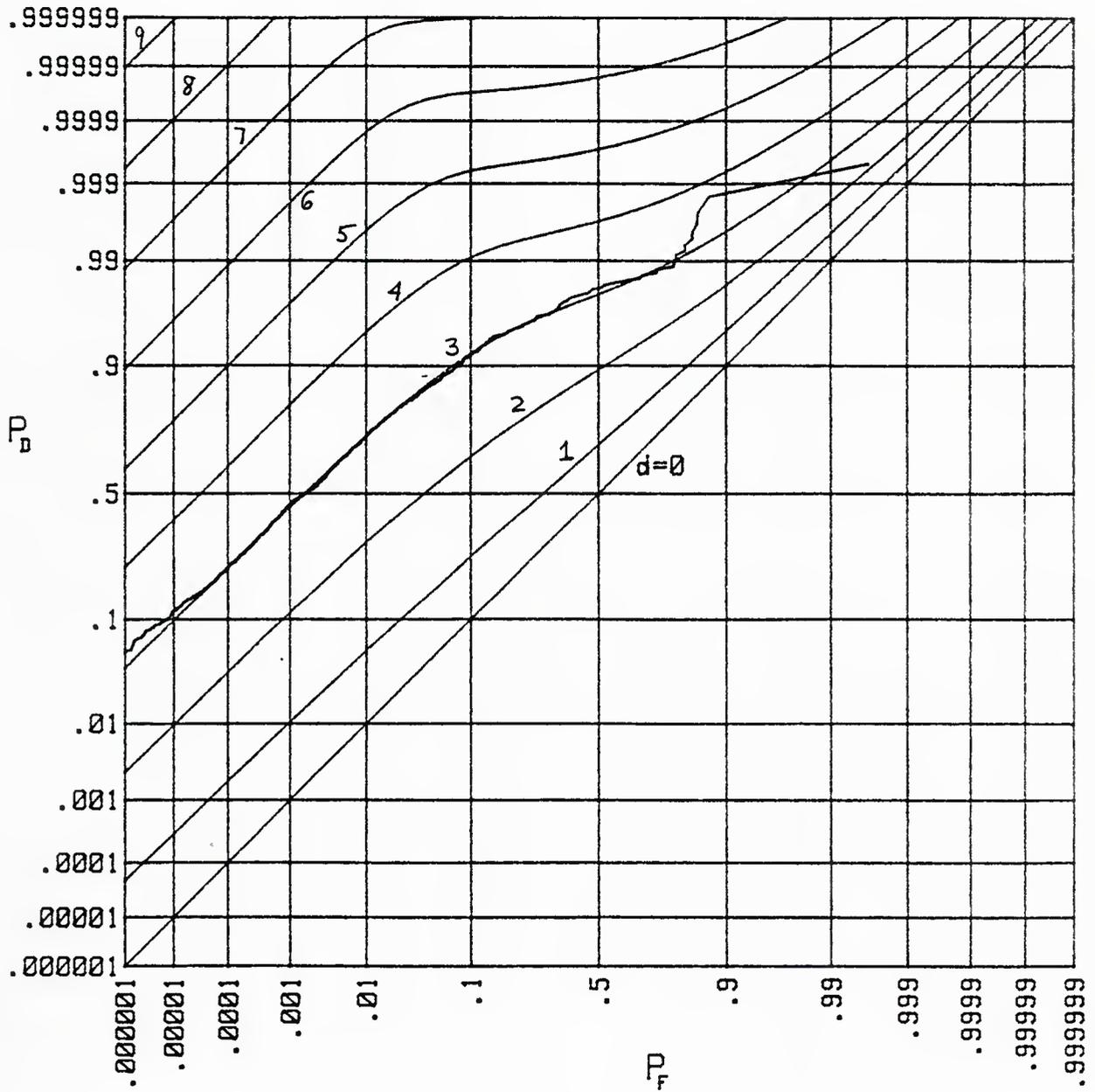
Figure 36. Operating Characteristics for N=4, K=1, M=1
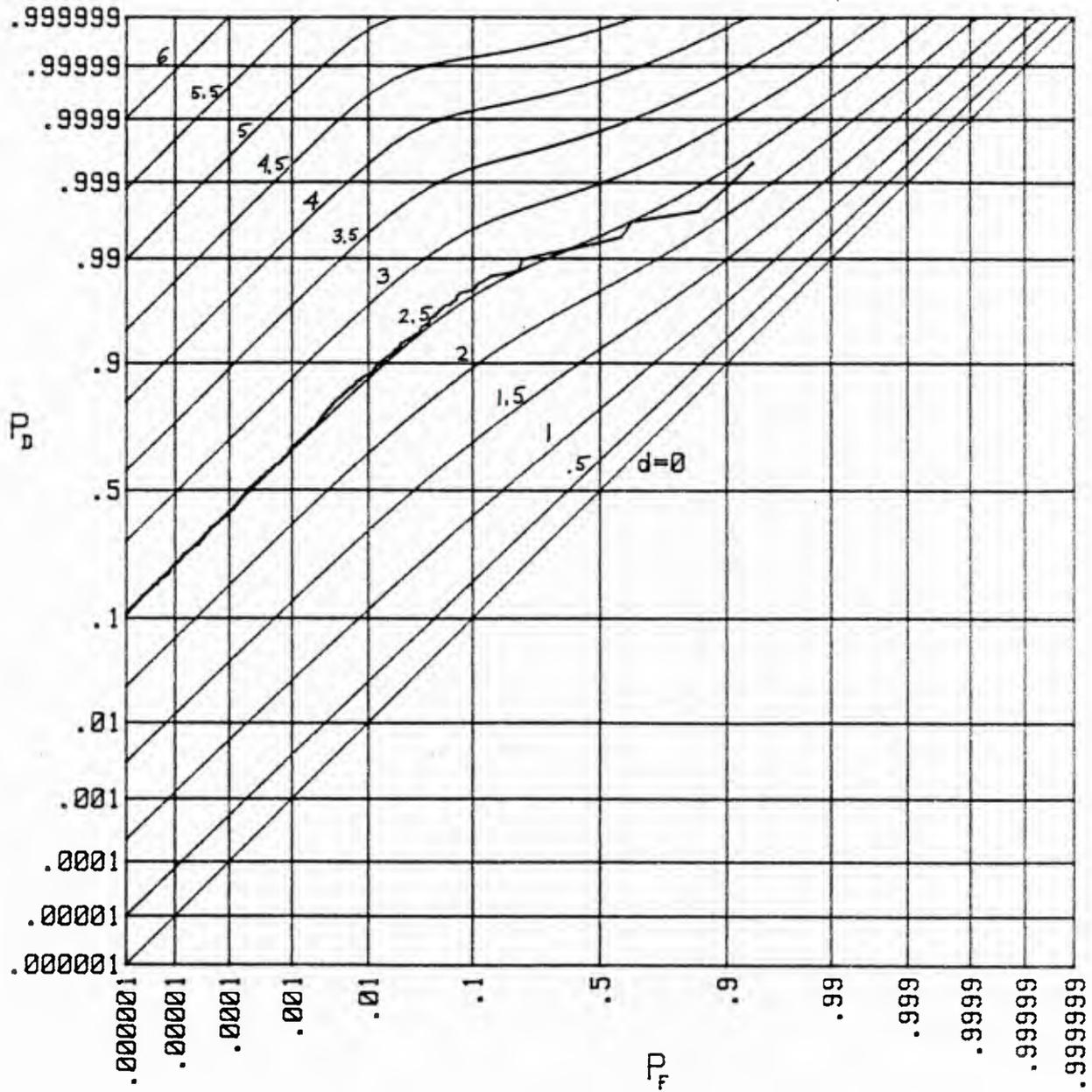
Figure 37.   Operating Characteristics for N=4, K=2, M=2
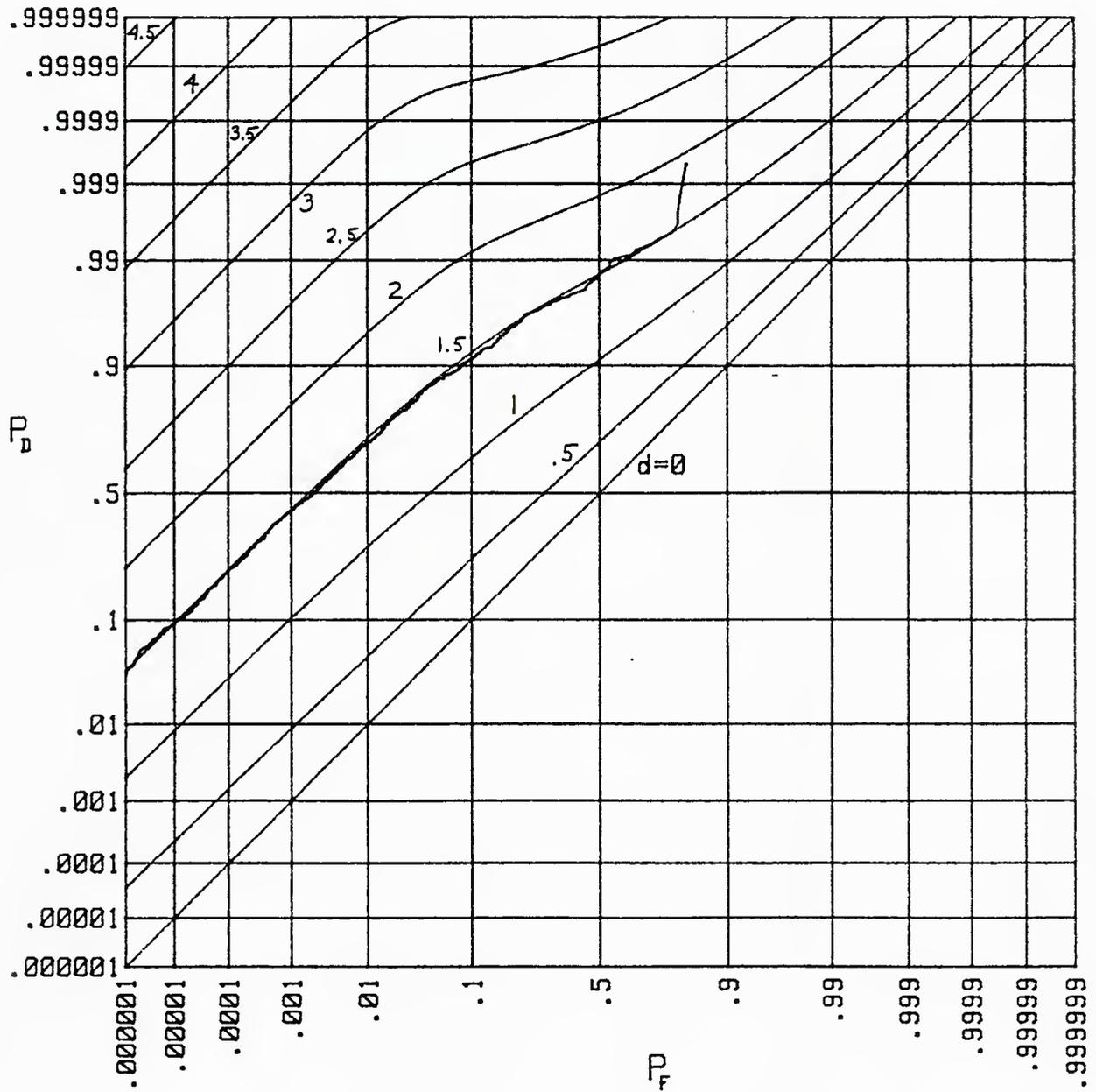
Figure 38.   Operating Characteristics for N=4, K=4, M=4
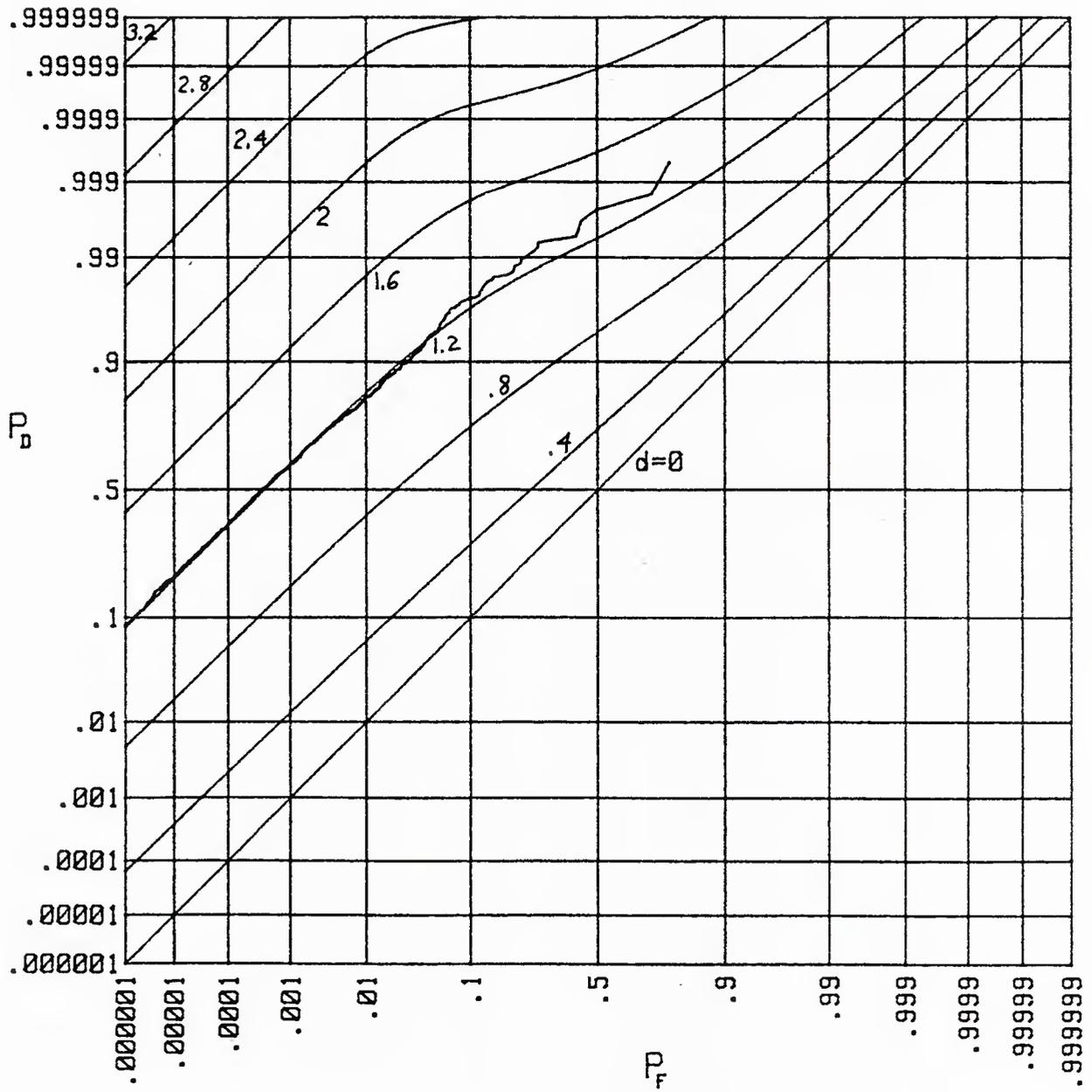
Figure 39. Operating Characteristics for N=4, K=8, M=8

APPENDIX A. CHARACTERISTIC FUNCTION OF DECISION VARIABLE

With reference to figure 1, let $\underset{\sim}{\ell} = (\ell_0, \ell_1, \ldots, \ell_{M-1})$ be
the time sequence of values of $\ell(t)$ at the output of the Greatest-Of device
leading to a particular value of the decision variable $w(t)$ at some arbitrary
time instant. Then the conditional characteristic function of $w(t)$ is, since
the inputs $\{x_n\}$ to the system are all statistically independent in time,

$$f_w(\xi|\underset{\sim}{\ell}) = \prod_{j=0}^{M-1} f_{\ell_j}(\xi) \quad , \qquad \text{(A-1)}$$

where the individual characteristic functions in (A-1) are

$$f_{\ell_j}(\xi) = \begin{cases} f_x^{(1)}(\xi) & \text{if signal is present in channel no. } \ell_j \\ f_x^{(0)}(\xi) & \text{if signal is absent from channel no. } \ell_j \end{cases} \quad . \qquad \text{(A-2)}$$

Recall that $f_x^{(h)}(\xi)$ is the characteristic function of a single channel
input under hypothesis h; h = 0 or 1.

H_0: No signal present in any input channel

The time sequence of selector outputs $\{v(t)\}$ is one of statistically
independent random variables, regardless of whether sequence $\{\ell(t)\}$ remains at
the same value (channel number) for several successive time instants, as shown
in figures 2 and 3. So the conditional characteristic function of $w(t)$ is,
under H_0,

$$f_w^{(0)}(\xi|\underset{\sim}{\ell}) = \left[f_x^{(0)}(\xi)\right]^M \quad , \qquad \text{(A-3)}$$

independent of the particular sequence $\underset{\sim}{\ell}$. Therefore the unconditional
characteristic function of $w(t)$ is

$$f_w^{(0)}(\xi) = \left[f_x^{(0)}(\xi)\right]^M \quad . \qquad \text{(A-4)}$$

Thus, the cumulative distribution function of decision variable $w(t)$ under H_0 can be easily evaluated [10,11] and the false alarm probability can be accurately calculated. The false alarm probability is independent of the number of input channels, N, and the amount of pre-averaging, K. The false alarm probability depends solely on the noise-only statistics at a single channel input, the amount of post-averaging, and the threshold. The threshold need not be increased as N is increased, in order to maintain a fixed false alarm probability. Arbitrary input statistics of $\{x_n(t)\}$ are allowed, except for the independent identically-distributed assumption.

H_1: Signal present in one unknown input channel

The conditional characteristic function of $w(t)$, using the independence in time of $\{x_n(t)\}$, is now given by

$$f_w^{(1)}(\xi|\underset{\sim}{\ell}) = \left[f_x^{(1)}(\xi)\right]^T \left[f_x^{(0)}(\xi)\right]^{M-T} \quad , \tag{A-5}$$

where T is the number of times that the values of the sequence $\underset{\sim}{\ell}$ correspond with the correct (unknown) signal channel. Thus the unconditional characteristic function of the decision variable $w(t)$ is

$$f_w^{(1)}(\xi) = \sum_{T=0}^{M} \text{Prob}(T) \; f_w^{(1)}(\xi|\underset{\sim}{\ell})$$

$$= \sum_{T=0}^{M} \text{Prob}(T) \; \left[f_x^{(1)}(\xi)\right]^T \left[f_x^{(0)}(\xi)\right]^{M-T} \quad , \tag{A-6}$$

where $\text{Prob}(T)$ is the probability of exactly T correspondences of $\underset{\sim}{\ell}$ with the correct signal channel.

Unfortunately, evaluation of $\text{Prob}(T)$ is very difficult. For example, using the independence of the channel inputs, the simplest case of $T=M$ yields

$$\text{Prob}(T{=}M) = \int\limits_{-\infty}^{+\infty} du_1 \cdots \int\limits_{-\infty}^{+\infty} du_M\, p_a^{(1)}(u_1, \ldots, u_M)$$

$$*\left[\int\limits_{-\infty}^{u_1} dv_1 \cdots \int\limits_{-\infty}^{u_M} dv_M\, p_a^{(0)}(v_1, \ldots, v_M)\right]^{N-1} , \qquad (A\text{-}7)$$

where $p_a^{(h)}(\ldots)$ is the joint probability density function of M adjacent time samples of one channel input to the Greatest-Of. However, probability density functions $p_a^{(0)}(v_1, \ldots, v_M)$ and $p_a^{(1)}(u_1, \ldots, u_M)$ cannot be factored, because neighboring time samples of $a_n(t)$, at a particular pre-averager output, are highly dependent. (If $\{a_n(t)\}$ samples were independent in time, as for K=1, this would simplify to

$$\text{Prob}(T{=}M) = \left\{\int\limits_{-\infty}^{+\infty} du\, p_a^{(1)}(u)\left[P_a^{(0)}(u)\right]^{N-1}\right\}^M \quad . \;) \qquad (A\text{-}8)$$

So we must resort to an approximate analysis under H_1 in order to determine $\text{Prob}(T)$.

Approximate Analysis under H_1

The plots in figures 2 and 3, of $\{a_n(t)\}$ vs. time sample t, reveal that a particular pre-averager output can dominate for many adjacent time samples (even if there is no signal present). Thus a typical plot of $\ell(t)$ appears as in figure A-1.
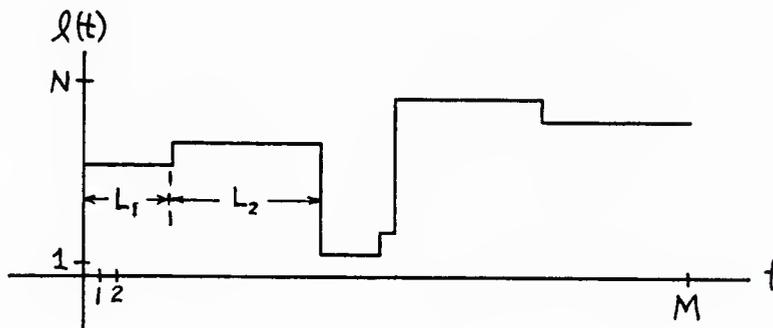
Figure A-1. Sample Plot of $\ell(t)$

The effective number of "independent samples", I_ℓ, of $\ell(t)$ in an interval of length M can be considerably less than M; it depends critically on K, the amount of pre-averaging. And since $\ell(t)$ depends directly on the pre-averager outputs $\{a_n(t)\}_1^N$, it appears that the effective number of independent

samples, I_a, of one of the pre-averager outputs in an interval M should be approximately the same as I_ℓ. This assumption precludes the dependence of I_ℓ on N, the number of channels; how good this presumption is must be borne out by comparison of analytical results with simulation.

The effective number, for a time sequence $a_n(t)$ observed for an interval M, is given by [12, eq. B-12]

$$I_a = \frac{M}{\sum_{m=-M}^{M} \left(1 - \frac{|m|}{M}\right) \rho_a(m)} \quad , \tag{A-9}$$

where $\rho_a(m)$ is the normalized covariance of stationary process $a_n(t)$; i.e., for any n,

$$\rho_a(m) = \frac{E\left\{\tilde{a}_n(t)\ \tilde{a}_n(t-m)\right\}}{E\left\{\tilde{a}_n^2(t)\right\}} \quad , \tag{A-10}$$

where $\tilde{a}_n(t)$ is the ac component of $a_n(t)$. But since $a_n(t)$ is the sum of K independent time samples of $x_n(t)$, there follows

$$\rho_a(m) = \begin{cases} 1 - \dfrac{|m|}{K} & \text{for} \quad |m| < K \\ 0 & \text{otherwise} \end{cases} \quad , \tag{A-11}$$

regardless of whether signal is present or not in the channel under consideration. Substitution of (A-11) in (A-9) yields

$$I_\ell \cong I_a = \begin{cases} \dfrac{3M^2}{3KM+1-K^2} & \text{for} \quad K \leq M \\ \dfrac{3KM}{3KM+1-M^2} & \text{for} \quad M \leq K \end{cases} \quad . \tag{A-12}$$

A reasonable approximation to (A-12) is $I_\ell \cong M/K$ for $M \geq K$; however, we use the more accurate result (A-12) in the following.

Now let probability

$$Q_a \equiv \text{Prob} \left\{ \text{signal channel dominates Greatest-Of on one trial} \right\}. \qquad \text{(A-13)}$$

Using the statistical independence between input channels, this quantity is given by

$$Q_a = \int du \; p_a^{(1)}(u) \left[P_a^{(0)}(u) \right]^{N-1} \qquad ; \qquad \text{(A-14)}$$

this is the special case of (A-7) or (A-8) for T=1. Then in an interval of length M, where there are I_ℓ effectively-independent samples (as given by (A-12)), the probability that the signal channel dominates the Greatest-Of exactly n times is

$$\text{Prob}\{n; \; I_\ell\} = \binom{I_\ell}{n} Q_a^n (1-Q_a)^{I_\ell - n} \qquad \text{for} \quad 0 \leq n \leq I_\ell \quad . \qquad \text{(A-15)}$$

We are presuming I_ℓ to be an integer for now; we shall alleviate this restriction shortly.

Now define

$$L = \frac{M}{I_\ell} \quad . \qquad \text{(A-16)}$$

This is the average number of time samples, in interval length M, for which a particular channel dominates the Greatest-Of. In terms of figure A-1, L is the average of lengths L_1, L_2, ... We then approximate the actual $\ell(t)$ plot in figure A-1 by that in figure A-2; there are I_ℓ independent steps in interval M, each of length L.
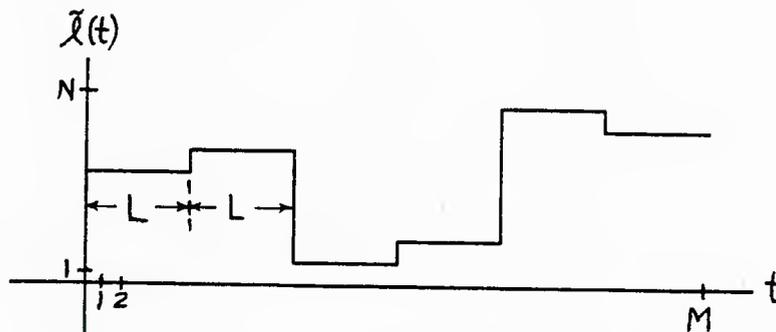
Figure A-2. Approximation $\tilde{\ell}(t)$ to $\ell(t)$

A-5

Under this approximation, the random variable T defined in (A-5)-(A-6) can only take on the values

$$T = 0 \quad \text{or} \quad L \quad \text{or} \quad 2L \quad \text{or} \quad \dots \quad \text{or} \quad M, \tag{A-17}$$

that is

$$T = nL \quad \text{for} \quad 0 \le n \le I_\ell \quad . \tag{A-18}$$

Therefore the conditional characteristic function (A-5) becomes

$$f_W^{(1)}(\xi|\underline{\ell}) = \left[f_X^{(1)}(\xi)\right]^{nL} \left[f_X^{(0)}(\xi)\right]^{M-nL} \quad \text{for} \quad 0 \le n \le I_\ell \quad . \tag{A-19}$$

Employment of (A-15) now yields the unconditional characteristic function of decision variable w(t) under H_1 as

$$f_W^{(1)}(\xi) = \sum_{n=0}^{I_\ell} \text{Prob}\{n; I_\ell\} \; f_W^{(1)}(\xi|\underline{\ell})$$

$$= \sum_{n=0}^{I_\ell} \binom{I_\ell}{n} Q_a^n \, (1-Q_a)^{I_\ell-n} \left[f_X^{(1)}(\xi)\right]^{nL} \left[f_X^{(0)}(\xi)\right]^{M-nL}$$

$$= \left\{ Q_a \left[f_X^{(1)}(\xi)\right]^L + (1-Q_a) \left[f_X^{(0)}(\xi)\right]^L \right\}^{I_\ell} \quad . \tag{A-20}$$

This is the main result of this appendix.

To review: the characteristic functions $f_X^{(h)}(\xi)$ are those for the system inputs $\{x_n(t)\}$ under hypotheses h = 0 and 1; I_ℓ is given by (A-12) for various amounts of pre-averaging K and post-averaging M; L is given by (A-16); and Q_a is given by (A-14), where $p_a^{(h)}(u)$ is the first-order probability density function of a pre-averager output under hypothesis h. Since $a_n(t)$ is the sum of K independent random variables, we have characteristic functions

$$f_a^{(h)}(\varsigma) = \left[f_x^{(h)}(\varsigma)\right]^K \quad \text{for} \quad h = 0 \text{ and } 1, \tag{A-21}$$

which can be used to determine probability density functions $p_a^{(h)}(u)$. The fundamental inputs required are N, K, M, $f_x^{(1)}(\varsigma)$, and $f_x^{(0)}(\varsigma)$.

For Q_a near 1, it is necessary to numerically evaluate $1-Q_a$ separately from (A-14); via physical reasoning or an integration by parts on (A-14), there follows

$$1-Q_a = (N-1) \int du \, P_a^{(1)}(u) \, p_a^{(0)}(u) \left[P_a^{(0)}(u)\right]^{N-2} . \tag{A-22}$$

We employ both (A-14) and (A-21) in the program that follows.

Special Cases of Characteristic Function $f_w^{(1)}(\varsigma)$

By taking the derivative of (A-20) with respect to ς and then setting $\varsigma=0$, we find the mean value of $w(t)$ under H_1 as

$$\lambda_w^{(1)}(1) = M\left[Q_a \, \lambda_x^{(1)}(1) + (1-Q_a) \, \lambda_x^{(0)}(1)\right] , \tag{A-23}$$

where $\lambda_r^{(h)}(n)$ is the n-th cumulant of random variable r under hypothesis h. Although (A-23) is deduced from the approximate characteristic function $f_w^{(1)}(\varsigma)$ given in (A-20), this result is in fact exact, as may be easily verified by an independent analysis.

From (A-20), we also can extract the variance of decision variable $w(t)$; there follows

$$\lambda_w^{(1)}(2) = M\left[Q_a \, \lambda_x^{(1)}(2) + (1-Q_a) \, \lambda_x^{(0)}(2) + LQ_a(1-Q_a)\left\{\lambda_x^{(1)}(1) - \lambda_x^{(0)}(1)\right\}^2\right] . \tag{A-24}$$

Since the exact calculation of the variance of $w(t)$ is very difficult, we have no check on this result.

As the input signal strength approaches zero, $f_x^{(1)}(\varsigma) \to f_x^{(0)}(\varsigma)$, and

(A-20) yields $f_W^{(1)}(\xi) \rightarrow \left[f_X^{(0)}(\xi) \right]^M$, which checks the H_0 result in (A-4).

The following are a series of checks, that are obviously correct. If N=1, then $Q_a=1$ from (A-14), and*

$$f_W^{(1)}(\xi) = \left[f_X^{(1)}(\xi) \right]^M \quad . \tag{A-25}$$

On the other hand, if M=1, then $I_\ell=1$ from (A-21), and

$$f_W^{(1)}(\xi) = Q_a \, f_X^{(1)}(\xi) + (1-Q_a) \, f_X^{(0)}(\xi) \quad ; \tag{A-26}$$

then probability density function

$$p_W^{(1)}(u) = Q_a \, p_X^{(1)}(u) + (1-Q_a) \, p_X^{(0)}(u) \tag{A-27}$$

giving detection probability

$$P_D = \text{Prob}\left\{ w > \Lambda \,\middle|\, H_1 \right\} = \int_\Lambda^\infty du \, p_W^{(1)}(u)$$

$$= Q_a \left[1 - P_X^{(1)}(\Lambda) \right] + (1-Q_a) \left[1 - P_X^{(0)}(\Lambda) \right]$$

$$= 1 - Q_a \, P_X^{(1)}(\Lambda) - (1-Q_a) \, P_X^{(0)}(\Lambda) \quad , \tag{A-28}$$

where Λ is the value of the threshold in figure 1. Finally, if K=1, then $I_\ell = M$ from (A-12), and

$$f_W^{(1)}(\xi) = \left[Q_a \, f_X^{(1)}(\xi) + (1-Q_a) \, f_X^{(0)}(\xi) \right]^M \quad ; \tag{A-29}$$

this result and Q_a in (A-14) are exact in this case, as noted in (A-8).

---

* For the Gaussian example in (C-1), we have the exact result for N=1,

$$\text{Prob}\left\{ w > \Lambda \,\middle|\, H_h \right\} = \Phi\left(\frac{M\mu_X^{(h)} - \Lambda}{\sqrt{M} \, \sigma_X} \right) \quad .$$

APPENDIX B. NON-INTEGER POWER OF A CHARACTERISTIC FUNCTION

The quantities L and I_ℓ in (A-20) will generally not be integer. So we may encounter the problem that even if $f(\xi)$ is a legal characteristic function, $[f(\xi)]^\nu$ may not be, for non-integer ν. (If it is a legal characteristic function, we ignore the following procedure.) We circumvent the illegal characteristic function problem by approximating

$$[f(\xi)]^\nu \quad \text{by legal characteristic function} \quad [f(\alpha\xi)]^I \, f(\beta\xi) \quad , \quad \text{(B-1)}$$

where I is the integer part of ν, and α, β are determined by equating the first two cumulants of these two expressions in (B-1). There follows

$$\nu = I\alpha + \beta, \quad \nu = I\alpha^2 + \beta^2 \quad . \tag{B-2}$$

The solution of these two equations is

$$\alpha = \frac{\nu I + (\nu I(1+I-\nu))^{1/2}}{I(I+1)} \quad , \quad \beta = \nu - I\alpha \quad . \tag{B-3}$$

Observe that I, α, β are functions solely of ν, and are independent of the statistics $f(\xi)$. We find generally that $\alpha \geq 1$ and $0 \leq \beta < 1$; observe in figure B-1 that α approaches 1 as ν increases.

For ν an integer, there follows $I = \nu$, $\alpha = 1$, $\beta = 0$, as expected. Also, for a Gaussian random variable, replacement (B-1) is in fact exact, as both functions yield

$$\exp(i\xi\mu\nu - \frac{1}{2}\xi^2 \sigma^2 \nu) \quad \text{for any } \nu \quad , \tag{B-4}$$

where μ and σ^2 are the mean and variance of the Gaussian random variable.

We now apply this procedure to (A-20). Define, as in (B-3),
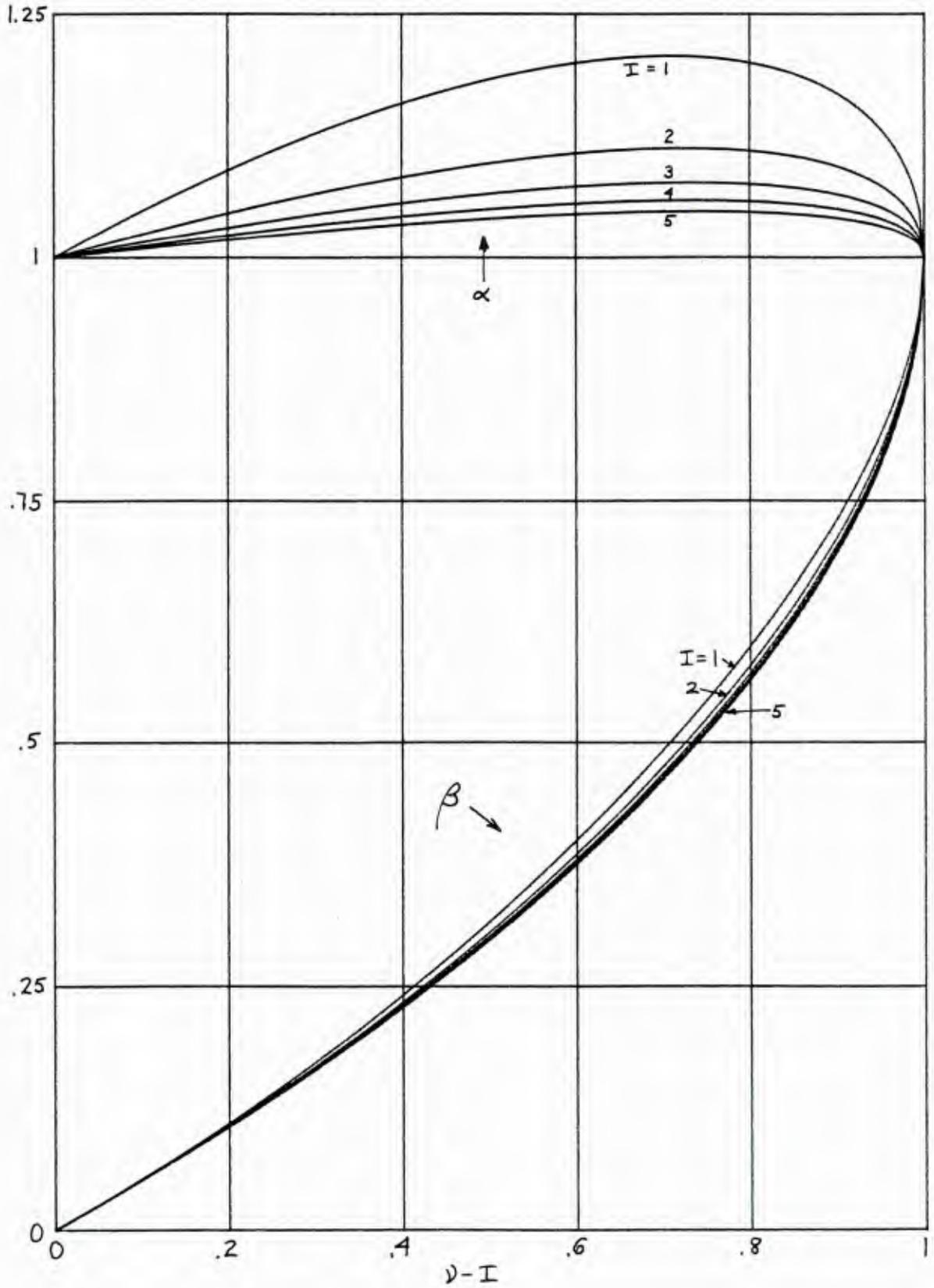
Figure B-1. Parameters of Approximation (B-1)

$$I_1 = \text{INT}(L) \text{ with attendant } \alpha_1, \beta_1;$$

$$I_2 = \text{INT}(I_\ell) \text{ with attendant } \alpha_2, \beta_2. \tag{B-5}$$

Then the interior terms of (A-20) are replaced by

$$\left[f_X^{(h)}(\xi)\right]^L \cong \left[f_X^{(h)}(\alpha_1\xi)\right]^{I_1} f_X^{(h)}(\beta_1\xi) \equiv f_1^{(h)}(\xi) \quad . \tag{B-6}$$

The bracketed quantity in (A-20) then becomes

$$Q_a f_1^{(1)}(\xi) + (1-Q_a) f_1^{(0)}(\xi) \equiv f_1(\xi) \quad . \tag{B-7}$$

Finally, (A-20) becomes

$$f_W^{(1)}(\xi) = \left[f_1(\xi)\right]^{I_\ell} = [f_1(\alpha_2\xi)]^{I_2} f_1(\beta_2\xi)$$

$$= \left[Q_a f_1^{(1)}(\alpha_2\xi) + (1-Q_a) f_1^{(0)}(\alpha_2\xi)\right]^{I_2} \left[Q_a f_1^{(1)}(\beta_2\xi) + (1-Q_a) f_1^{(0)}(\beta_2\xi)\right] \tag{B-8}$$

$$= \left\{Q_a \left[f_X^{(1)}(\alpha_1\alpha_2\xi)\right]^{I_1} f_X^{(1)}(\beta_1\alpha_2\xi) + (1-Q_a) \left[f_X^{(0)}(\alpha_1\alpha_2\xi)\right]^{I_1} f_X^{(0)}(\beta_1\alpha_2\xi)\right\}^{I_2}$$

$$\star \left\{Q_a \left[f_X^{(1)}(\alpha_1\beta_2\xi)\right]^{I_1} f_X^{(1)}(\beta_1\beta_2\xi) + (1-Q_a) \left[f_X^{(0)}(\alpha_1\beta_2\xi)\right]^{I_1} f_X^{(0)}(\beta_1\beta_2\xi)\right\} \tag{B-9}$$

in terms of the input characteristic functions $f_X^{(h)}$. This form is useful when $f_X^{(h)}$ are available numerically. An alternative form is available from (B-8) as

$$f_W^{(1)}(\xi) = \left[Q_a f_1^{(1)}(\beta_2\xi) + (1-Q_a) f_1^{(0)}(\beta_2\xi)\right]$$

$$\star \sum_{n=0}^{I_2} \binom{I_2}{n} Q_a^n (1-Q_a)^{I_2-n} \left[f_1^{(1)}(\alpha_2\xi)\right]^n \left[f_1^{(0)}(\alpha_2\xi)\right]^{I_2-n} \quad . \tag{B-10}$$

This is useful if the products of characteristic functions can be handled easily or have easily-recognized probability density functions.

APPENDIX C. GAUSSIAN INPUT EXAMPLE

The inputs $\{x_n(t)\}$ to the system of figure 1 are Gaussian with characteristic function

$$f_x^{(h)}(\xi) = \exp(i\xi\mu_x^{(h)} - \frac{1}{2}\xi^2 \sigma_x^2) \quad \text{for} \quad h = 0 \text{ and } 1 \quad . \quad \text{(C-1)}$$

The input means are $\mu_x^{(0)}$ and $\mu_x^{(1)}$ under hypothesis H_0 and H_1 respectively, while the input variance is common value σ_x^2 under both hypotheses.

Since $a_n(t)$ is the sum of K statistically independent random variables, the characteristic function of $a_n(t)$ is

$$f_a^{(h)}(\xi) = \exp(i\xi\mu_x^{(h)}K - \frac{1}{2}\xi^2 \sigma_x^2 K) \quad . \quad \text{(C-2)}$$

The corresponding probability density function and distribution function are

$$p_a^{(h)}(u) = \frac{1}{\sqrt{K}\,\sigma_x} \; \phi\left(\frac{u-K\mu_x^{(h)}}{\sqrt{K}\,\sigma_x}\right) \quad \text{(C-3)}$$

and

$$P_a^{(h)}(u) = \Phi\left(\frac{u-K\mu_x^{(h)}}{\sqrt{K}\,\sigma_x}\right) \quad , \quad \text{(C-4)}$$

where

$$\phi(u) \equiv (2\pi)^{-1/2} \exp(-u^2/2), \quad \Phi(u) \equiv \int_{-\infty}^{u} dv \; \phi(v) \quad . \quad \text{(C-5)}$$

Substitution of (C-3) and (C-4) in (A-14) and (A-22) yields

$$Q_a = q \; (\sqrt{K}\, d, \; N-1),$$

$$1-Q_a = q_1 \; (\sqrt{K}\, d, \; N-1), \quad \text{(C-6)}$$

where

$$q(c,m) \equiv \int dx \; \phi(x) \; [\Phi(x+c)]^m \quad ,$$

$$q_1(c,m) \equiv m \int dx \; \phi(x) \; [\Phi(x)]^{m-1} \; \Phi(x-c) \quad , \tag{C-7}$$

and

$$d \equiv \frac{\mu_x^{(1)} - \mu_x^{(0)}}{\sigma_x} \quad . \tag{C-8}$$

The quantity d is called the input deflection. Special cases of (C-7) are $q(c,0) = 1$ and $q(c,1) = \Phi(c/\sqrt{2})$.

The characteristic function of decision variable w(t) under H_0 is immediately available by substitution of (C-1) in (A-4):

$$f_w^{(0)}(\xi) = \exp(i\xi\mu_x^{(0)} M - \tfrac{1}{2}\xi^2 \sigma_x^2 M) \quad . \tag{C-9}$$

Then the probability density function of w under H_0 is

$$p_w^{(0)}(u) = \frac{1}{\sqrt{M}\,\sigma_x} \; \phi\!\left(\frac{u - M\mu_x^{(0)}}{\sqrt{M}\,\sigma_x}\right) \tag{C-10}$$

and the false alarm probability is

$$P_F = \text{Prob}\left\{w > \Lambda \,\middle|\, H_0\right\} = \int_{\Lambda}^{\infty} du \; p_w^{(0)}(u) = \Phi(\gamma) \quad , \tag{C-11}$$

where Λ is the threshold in figure 1, and

$$\gamma \equiv \frac{M\mu_x^{(0)} - \Lambda}{\sqrt{M}\,\sigma_x} \quad . \tag{C-12}$$

Under H_1, we employ (C-1) in (B-6) and get

$$f_1^{(h)}(\xi) = \left[f_x^{(h)}(\xi)\right]^L = \exp(i\xi\mu_x^{(h)} L - \tfrac{1}{2}\xi^2 \sigma_x^2 L) \quad , \tag{C-13}$$

which is exact for any L (as noted in (B-4)); there is no need here to take the integer part of L, as was done in (B-5) and (B-9) for the general case.

C-2

We now use (C-13) in (B-10) and find characteristic function

$$f_w^{(1)}(\xi) = \sum_{n=0}^{I_2} \binom{I_2}{n} Q_a^n (1-Q_a)^{I_2-n}$$

$$*\left[Q_a \exp\left(i\xi m_1(n) - \frac{1}{2}\xi^2 \sigma^2\right) + (1-Q_a) \exp\left(i\xi m_2(n) - \frac{1}{2}\xi^2 \sigma^2\right)\right] \quad , \quad (C-14)$$

where

$$m_1(n) \equiv \mu_x^{(0)} M + \left(\mu_x^{(1)} - \mu_x^{(0)}\right)\left(\beta_2 + \alpha_2 n\right) L,$$

$$m_2(n) \equiv \mu_x^{(0)} M + \left(\mu_x^{(1)} - \mu_x^{(0)}\right) \alpha_2 n L,$$

$$\sigma^2 \equiv \sigma_x^2 M \quad . \tag{C-15}$$

Here we have used (B-2) and (B-5) in the form

$$I_\ell = I_2 \alpha_2 + \beta_2,$$

$$I_\ell = I_2 \alpha_2^2 + \beta_2^2, \tag{C-16}$$

and (A-16) in the form $M = LI_\ell$.

The probability density function of decision variable $w(t)$ under H_1 is then

$$p_w^{(1)}(u) = \sum_{n=0}^{I_2} \binom{I_2}{n} Q_a^n (1-Q_a)^{I_2-n}$$

$$*\left[Q_a \frac{1}{\sigma} \phi\left(\frac{u - m_1(n)}{\sigma}\right) + (1-Q_a) \frac{1}{\sigma} \phi\left(\frac{u - m_2(n)}{\sigma}\right)\right] \quad . \tag{C-17}$$

Finally the detection probability is

$$P_D = \text{Prob}\left\{w > \lambda \big| H_1\right\} = \int_{\lambda}^{\infty} du \; p_w^{(1)}(u)$$

$$= \sum_{n=0}^{I_2} \binom{I_2}{n} Q_a^n (1-Q_a)^{I_2-n} \left[Q_a \Phi\left(\frac{m_1(n) - \lambda}{\sigma}\right) + (1-Q_a) \; \Phi\left(\frac{m_2(n) - \lambda}{\sigma}\right) \right] \quad . \qquad (C\text{-}18)$$

In order to put this result in a form consistent with that for the false alarm probability in (C-11), we use (C-12), (C-15), (C-8), and (A-16), to get

$$P_D = \sum_{n=0}^{I_2} \binom{I_2}{n} Q_a^n (1-Q_a)^{I_2-n}$$

$$* \left[Q_a \Phi\left(\gamma + d\sqrt{M} \; \frac{\beta_2 + \alpha_2 \, n}{I_\ell}\right) + (1-Q_a) \; \Phi\left(\gamma + d\sqrt{M} \; \frac{\alpha_2 n}{I_\ell}\right) \right] . \qquad (C\text{-}19)$$

To review: d is the input deflection as given by (C-8); M is the amount of post-averaging; I_ℓ is the effective number of independent samples of ℓ, as given by (A-12) in terms of K and M; I_2 is the integer part of I_ℓ; α_2 and β_2 are the solutions of (C-16) and can be found via (B-3); Q_a and $1-Q_a$ are probabilities given by (C-6) and (C-7); and γ is a modified threshold given by (C-12) such that the false alarm probability is simply

$$P_F = \Phi(\gamma) \quad . \qquad (C\text{-}20)$$

As $\mu_x^{(1)} \to \mu_x^{(0)}$, then $d \to 0$, and (C-19) tends to (C-20) as it should. For $N=1$, then $Q_a=1$ and (C-19) reduces to $P_D = \Phi(\gamma + d\sqrt{M})$. Equations (C-19) and (C-20) for the detection probability and false alarm probability are the end results of this appendix. The fundamental inputs are N, K, M, d.

Simulation

Since the absolute scales of $\mu_x^{(1)}$, $\mu_x^{(0)}$, and σ_x do not enter separately into the detection probability and false alarm probability, but only through the combination

$$d = \frac{\mu_x^{(1)} - \mu_x^{(0)}}{\sigma_x} \quad , \qquad (C\text{-}21)$$

we let $\mu_X^{(0)} = 0$ and $\sigma_X = 1$ in a simulation. Then from (C-11) and (C-12),

$$P_F = \bar{\Phi}(-\Lambda/\sqrt{M});\qquad\qquad\text{(C-22)}$$

this result is exact, so there is no need to simulate the false alarm case. Under H_1, suppose we make a total of T_t trials, getting decision variables $\{w_k\}_1^{T_t}$, ordered such that $w_1 \leq w_2 \leq \cdots \leq w_{T_t}$. Then the detection probability and false alarm probability are given by

$$\left.\begin{aligned}
P_D &= \text{Prob}\left\{w > \Lambda \middle| H_1\right\} \equiv 1 - \frac{k - \frac{1}{2}}{T_t}\\[2em]
P_F &= \text{Prob}\left\{w > \Lambda \middle| H_0\right\} = \bar{\Phi}(-w_k/\sqrt{M})
\end{aligned}\right\}\quad \text{at } \Lambda = w_k \text{ for } 1 \leq k \leq T_t \ . \qquad\text{(C-23)}$$

Sample simulation results for various values of d $\left(= \mu_X^{(1)}\right)$ are superposed on the analytical results, given by (C-19) and (C-20), in figures 4 through 35.

APPENDIX D.   PROGRAM FOR OPERATING CHARACTERISTICS

Since Q_a and $1-Q_a$ are evaluated separately according to (C-6) and (C-7), a partial check on accuracy is furnished by printing out their sum. Also, the array F(*) holds the binomial terms in the upper line of (C-19); its sum should be 1 and is also printed out.

```
10      T$="Figure 39.   Operating Characteristics"
20      N=4                     ! NUMBER OF INPUT CHANNELS
30      K=8                     ! LENGTH OF PRE-AVERAGERS
40      M=8                     ! LENGTH OF POST-AVERAGER
50      Dmax=4                  ! MAXIMUM VALUE FOR DEFLECTION d
60      Dinc=.4                 ! INCREMENT IN d
70      Dsim=1.2                ! VALUE OF d IN SIMULATION
80      Tt=1000                 ! TOTAL NUMBER OF TRIALS
90      OUTPUT 0;"NUMBER OF INPUT CHANNELS      N =";N
100     OUTPUT 0;"LENGTH OF PRE-AVERAGERS     K =";K
110     OUTPUT 0;"LENGTH OF POST-AVERAGER     M =";M
120     OUTPUT 0;"MAXIMUM VALUE FOR d         Dmax =";Dmax
130     OUTPUT 0;"INCREMENT IN d              Dinc =";Dinc
140     OUTPUT 0;"VALUE OF d IN SIMULATION  Dsim =";Dsim
150     OUTPUT 0;"TOTAL NUMBER OF TRIALS      Tt =";Tt
160     OUTPUT 0;""
170     INTEGER I,J,Ns,It
180     T=3*K*M
190     IF K<M THEN 220
200     I1=T/(T+1-M*M)
210     GOTO 230
220     I1=3*M*M/(T+1-K*K)
230     OUTPUT 0;"EFFECTIVE NUMBER OF INDEPENDENT SAMPLES I1 =";I1
240     OUTPUT 0;""
250     I2=INT(I1)
260     T=I2*I1
270     A2=(T+SQR(T*(1+I2-I1)))/(I2*(I2+1))
280     B2=I1-I2*A2
290     IF N=1 THEN 310
300     OUTPUT 0;"         Qa","        1-Qa","    Qa+(1-Qa)","     SUM F(*)"
310     REDIM F(0:I2)
320     DIM F(0:64)
330     PLOTTER IS "9872A"
340     OUTPUT 705;"VS5"
350     LIMIT 10,180,0,235                          ! LABELLING
360     MOVE 51,25
370     DIM T$[37]
380     N$=" for N="
390     K$=", K="
400     M$=", M="
410     B$=CHR$(8)
420     CSIZE 3,.5
430     LORG 5
440     LABEL T$;N$;B$;N;B$;K$;B$;K;B$;M$;B$;M
450     LIMIT 10,180,65,235                         ! GRID
460     LOCATE 12,98,12,98
470     DATA 1E-6,1E-5,1E-4,.001,.01,.1,.5
480     DATA .9,.99,.999,.9999,.99999,.999999
490     DIM P(1:13),Y(1:13)
500     READ P(*)
```

```
510    FOR I=1 TO 13
520    Y(I)=FNInvphi(P(I))         ! NORMAL PROBABILITY SCALES
530    NEXT I
540    S=Y(1)
550    B=Y(13)
560    SCALE S,B,S,B
570    FOR I=1 TO 13
580    MOVE S,Y(I)
590    DRAW B,Y(I)
600    NEXT I
610    FOR I=1 TO 13
620    MOVE Y(I),S
630    DRAW Y(I),B
640    NEXT I
650    MOVE S,S
660    DRAW B,B
670    CSIZE 3,.5
680    LORG 1
690    MOVE .4,.2
700    LABEL "d=0"
710    LORG 8
720    FOR I=1 TO 13
730    MOVE S+.1,Y(I)
740    LABEL P(I)
750    NEXT I
760    R=.5*(Y(7)+Y(8))
770    MOVE S-1,R
780    CSIZE 4,.5
790    LORG 8
800    LABEL "P"
810    MOVE S-1,R-.1
820    CSIZE 2,.5
830    LORG 6
840    LABEL "D"
850    CSIZE 3,.5
860    LDIR PI/2
870    LORG 8
880    FOR I=1 TO 13
890    MOVE Y(I),S+.05
900    LABEL P(I)
910    NEXT I
920    MOVE R,S-1
930    CSIZE 4,.5
940    LDIR 0
950    LORG 8
960    LABEL "P"
970    MOVE R,S-1.1
980    CSIZE 2,.5
990    LORG 6
1000   LABEL "F"
```

```
1010   Sk=SQR(K)                              ! ANALYTICAL RESULTS
1020   Sm=SQR(M)
1030   P13=P(13)
1040   REDIM Xp(1:Tt),Yp(1:Tt)
1050   DIM Xp(1:1000),Yp(1:1000)
1060   FOR Ds=Dinc TO Dmax STEP Dinc          ! DEFLECTION d IN (6)
1070   Dk=Ds*Sk
1080   Dm=Ds*Sm
1090   IF N>1 THEN 1130
1100   MOVE S,S+Dm
1110   DRAW B,B+Sm
1120   GOTO 1450
1130   Qa=FNQs(Dk,N-1)                        ! (C-6)
1140   Qa1=FNQs1(Dk,N-1)                      ! (C-6)
1150   F(0)=Qa1^I2
1160   FOR Ns=1 TO I2
1170   F(Ns)=F(Ns-1)*(I2+1-Ns)*Qa/(Ns*Qa1)   ! BINOMIAL TERMS IN (C-19)
1180   NEXT Ns
1190   OUTPUT 0;Qa,Qa1,Qa+Qa1,SUM(F)
1200   T=Dm/I1
1210   Ta2=T*A2
1220   Tb2=T*B2
1230   J=0
1240   FOR Gam=S TO B STEP (B-S)/50
1250   J=J+1
1260   Xp(J)=Gam
1270   G2=Gam+Tb2
1280   Pd=0
1290   FOR Ns=0 TO I2
1300   V=FNPhi(G2+Ta2*Ns)
1310   V1=FNPhi(Gam+Ta2*Ns)
1320   Pd=Pd+F(Ns)*(Qa*V+Qa1*V1)             ! (C-19)
1330   NEXT Ns
1340   D=Pd-.5
1350   IF ABS(D)<.49999999999 THEN 1380
1360   Yp(J)=100*SGN(D)
1370   GOTO 1390
1380   Yp(J)=FNInvphi(Pd)
1390   IF Pd>P13 THEN 1410
1400   NEXT Gam
1410   FOR I=1 TO J
1420   PLOT Xp(I),Yp(I)
1430   NEXT I
1440   PENUP
1450   NEXT Ds
1460   PENUP
```

```
1470   L4=-LOG(4)                                    ! SIMULATION RESULTS
1480   R=PI/180                                      ! SEED FOR RANDOM NUMBERS
1490   Km=K+M
1500   REDIM X(1:Km,1:N),W(1:Tt)
1510   DIM X(1:72,1:16),W(1:1000)
1520   FOR It=1 TO Tt
1530   FOR I=1 TO Km
1540   FOR Ns=1 TO N STEP 2
1550   H=1E-5*INT(1E5*R)
1560   L=R-H
1570   U=FRACT(31622000000*L)
1580   V=FRACT(777021*H)
1590   W=FRACT(777021*L)
1600   R=FRACT(U+V+W+1E-11)
1610   V1=R-.5
1620   H=1E-5*INT(1E5*R)
1630   L=R-H
1640   U=FRACT(31622000000*L)
1650   V=FRACT(777021*H)
1660   W=FRACT(777021*L)
1670   R=FRACT(U+V+W+1E-11)                          ! GENERATE TWO
1680   V2=R-.5                                       ! INDEPENDENT
1690   S2=V1*V1+V2*V2                                ! NORMALIZED
1700   IF S2>.25 THEN 1550                          ! GAUSSIAN
1710   Q2=(L4-LOG(S2))/S2                           ! RANDOM VARIABLES
1720   Q2=SQR(Q2+Q2)                                ! VIA
1730   X(I,Ns)=V1*Q2                                ! ACCEPTANCE-
1740   IF Ns+1>N THEN 1760                          ! REJECTION
1750   X(I,Ns+1)=V2*Q2                              ! METHOD
1760   NEXT Ns
1770   X(I,1)=X(I,1)+Dsim                           ! SIGNAL IN CHANNEL 1
1780   NEXT I
1790   Wt=0
1800   FOR I=K+1 TO Km
1810   Yt=-9E99
1820   FOR Ns=1 TO N
1830   A=0
1840   FOR J=1 TO K
1850   A=A+X(I-J,Ns)                                ! PRE-AVERAGER OUTPUT
1860   NEXT J
1870   IF Yt>A THEN 1900                           !    OR-ING OPERATION
1880   Yt=A                          ! LARGEST PRE-AVERAGER OUTPUT
1890   Lt=Ns         ! CHANNEL NUMBER OF LARGEST PRE-AVERAGER OUTPUT
1900   NEXT Ns
1910   Wt=Wt+X(I,Lt)                                ! POST-AVERAGER OUTPUT
1920   NEXT I
1930   W(It)=Wt
1940   NEXT It
1950   MAT SORT W
1960   F=1/Sm
1970   FOR It=1 TO Tt
1980   Pd=1-(It-.5)/Tt
1990   Yp(It)=FNInvphi(Pd)
2000   Xp(It)=T=-W(It)*F
2010   IF T<S THEN 2030
2020   NEXT It
2030   FOR I=1 TO MIN(It,Tt)
2040   PLOT Xp(I),Yp(I)
2050   NEXT I
2060   PENUP
2070   END
2080   !
```

```
2090    DEF FNQs(Cs,Ms)                              ! (C-7), LINE 1
2100    INTEGER K
2110    IF Ms=0 THEN RETURN 1
2120    IF Ms>1 THEN 2150
2130    V=FNPhi(Cs*.707106781187)
2140    RETURN V
2150    A=-7
2160    B=7
2170    S=(FNS(A,Cs,Ms)+FNS(B,Cs,Ms))*.5
2180    N=2
2190    H=(B-A)*.5
2200    F=(B-A)/3*.398942280401
2210    T=0
2220    FOR K=1 TO N-1 STEP 2
2230    T=T+FNS(A+H*K,Cs,Ms)
2240    NEXT K
2250    S=S+T
2260    Vo=V
2270    V=(S+T)*F
2280    PRINT V
2290    IF ABS(1-Vo/V)<=1E-9 THEN RETURN V
2300    N=N+N
2310    H=H*.5
2320    F=F*.5
2330    GOTO 2210
2340    FNEND
2350    !
2360    DEF FNQs1(Cs,Ms)                             ! (C-7), LINE 2
2370    INTEGER K
2380    IF Ms=0 THEN RETURN 0
2390    IF Ms>1 THEN 2420
2400    V=FNPhi(-Cs*.707106781187)
2410    RETURN V
2420    A=-7
2430    B=7
2440    S=(FNS1(A,Cs,Ms)+FNS1(B,Cs,Ms))*.5
2450    N=2
2460    H=(B-A)*.5
2470    F=(B-A)/3*Ms*.398942280401
2480    T=0
2490    FOR K=1 TO N-1 STEP 2
2500    T=T+FNS1(A+H*K,Cs,Ms)
2510    NEXT K
2520    S=S+T
2530    Vo=V
2540    V=(S+T)*F
2550    PRINT V
2560    Dv=ABS(V-Vo)
2570    IF (Dv<=1E-10) AND (Dv/V<=1E-6) THEN RETURN V
2580    N=N+N
2590    H=H*.5
2600    F=F*.5
2610    GOTO 2480
2620    FNEND
2630    !
```

```
2640   DEF FNS(X,Cs,Ms)                          ! INTEGRAND OF (C-7), LINE 1
2650   T=FNPhi(X+Cs)
2660   RETURN EXP(-.5*X*X)*T^Ms
2670   FNEND
2680   !
2690   DEF FNS1(X,Cs,Ms)                         ! INTEGRAND OF (C-7), LINE 2
2700   T=FNPhi(X)
2710   T1=FNPhi(X-Cs)
2720   RETURN EXP(-.5*X*X)*T^(Ms-1)*T1
2730   FNEND
2740   !
2750   DEF FNPhi(X)                              ! (C-5)
2760   INTEGER J
2770   IF ABS(X)>5.14 THEN 3030
2780   A=.282842712475*X
2790   C=COS(A)
2800   S=SIN(A)
2810   B=C+C
2820   A=B*C-1
2830   C=A*(1.2536751E-18+B*7.10005E-20+A*7.4517E-21)
2840   C=A*(1.533423425E-16+B*1.01649277E-17+C)
2850   C=A*(1.36760444757E-14+B*1.0601364636E-15+C)
2860   C=A*(8.89786526722E-13+B*8.06068838945E-14+C)
2870   C=A*(4.22616144318E-11+B*4.46968229249E-12+C)
2880   C=A*(1.46660614234E-9+B*1.80848587810E-10+C)
2890   C=A*(3.72252349369E-8+B*5.34275027603E-9+C)
2900   C=A*(6.91927520325E-7+B*1.15330990944E-7+C)
2910   C=A*(9.43281169838E-6+B*1.82066316364E-6+C)
2920   C=A*(9.44909268810E-5+B*2.10404583073E-5+C)
2930   C=A*(6.97183792408E-4+B*1.78228016255E-4+C)
2940   C=A*(3.80150767985E-3+B*1.10860645342E-3+C)
2950   C=A*(.0153985726157+B*.00507906961220+C)
2960   C=A*(.0467755234325+B*.0172439625887+C)
2970   C=A*(.108630245023+B*.0439773381941+C)
2980   C=A*(.201339747265+B*.0869894549959+C)
2990   C=A*(.330501521917+B*.144227226362+C)
3000   C=.703225002744+B*.247255168140+C
3010   Phi=.5+.0450158158079*X+.5*S*C
3020   RETURN Phi
3030   IF X>7 THEN RETURN 1
3040   N=MAX(6,INT(69/ABS(X)),INT(525/(X*X)))+1
3050   S=A=1
3060   B=1/X
3070   C=B*B
3080   FOR J=1 TO N
3090   A=(1-J-J)*A*C
3100   S=S+A
3110   NEXT J
3120   Phi=.398942280401*EXP(-.5*X*X)*ABS(B)*S
3130   IF X>0 THEN Phi=1-Phi
3140   RETURN Phi
3150   FNEND
3160   !
```

```
3170   DEF FNInvphi(Z)        !   INVERSE FUNCTION TO PHI IN (C-5)
3180   INTEGER N
3190   DIM T(0:20),A(0:20)
3200   DATA .992885376619,.120467516143,.0160781993421,.00268670443716
3210   DATA .49963473024E-3,.988982186E-4,.2039181276E-4,.432727162E-5
3220   DATA .93808141E-6,.20673472E-6,.461597E-7,.1041668E-7,.23715E-8
3230   DATA .54393E-9,.12555E-9,.2914E-10,.679E-11,.159E-11,.37E-12
3240   DATA .912158803418,-.0162662818677,.43355647295E-3,.21443857007E-3
3250   DATA .262575108E-5,-.302109105E-5,-.1240606E-7,.6240661E-7,-.54012E-9
3260   DATA -.142321E-8,.3438E-10,.3358E-10,-.146E-11,-.81E-12,.5E-13,.2E-13
3270   DATA .956679709020,-.0231070043091,-.00437423609751,-.57650342265E-3
3280   DATA -.1096102231E-4,.2510854702E-4,.1056233607E-4,.275441233E-5
3290   DATA .43248450E-6,-.2053034E-7,-.4389154E-7,-.1768401E-7,-.399129E-8
3300   DATA -.18693E-9,.27292E-9,.13282E-9,.3183E-10,.167E-11,-.204E-11
3310   DATA -.965E-12,-.22E-12
3320   X=Z+Z-1
3330   A=B=ABS(X)
3340   IF A>=.8 THEN B=SQR(-LOG(4*Z*(1-Z)))
3350   IF A<.8 THEN 3450
3360   IF A<.9975 THEN 3410
3370   Nmax=20
3380   RESTORE 3270
3390   Y=-.559457631330*B+2.28791571626
3400   GOTO 3480
3410   Nmax=15
3420   RESTORE 3240
3430   Y=-1.54881304237*B+2.56549012315
3440   GOTO 3480
3450   Nmax=18
3460   RESTORE 3200
3470   Y=12.5*Z*(Z-1)+2.125
3480   REDIM A(0:Nmax)
3490   READ A(*)
3500   Y2=Y+Y.
3510   T(0)=1
3520   T(1)=Y
3530   FOR N=2 TO Nmax
3540   T(N)=Y2*T(N-1)-T(N-2)
3550   NEXT N
3560   R=0
3570   FOR N=Nmax TO 0 STEP -1
3580   R=R+A(N)*T(N)
3590   NEXT N
3600   Invphi=SGN(X)*B*R*1.41421356237
3610   RETURN Invphi
3620   FNEND
```

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